hep-th/0212240
SNB/December/2002

GAUGE TRANSFORMATIONS, BRST COHOMOLOGY AND WIGNER'S LITTLE GROUP

R.P.Malik *

S. N. Bose National Centre for Basic Sciences,
Block-JD, Sector-III, Salt Lake, Calcutta-700 098, India

Abstract: We discuss the (dual-)gauge transformations and BRST cohomology for the two $(1 + 1)$-dimensional (2D) free Abelian one-form and four $(3 + 1)$-dimensional (4D) free Abelian 2-form gauge theories by exploiting the (co-)BRST symmetries (and their corresponding generators) for the Lagrangian densities of these theories. For the 4D free 2-form gauge theory, we show that the changes on the antisymmetric polarization tensor $e^{\mu\nu}(k)$ due to (i) the (dual-)gauge transformations corresponding to the internal symmetry group, and (ii) the translation subgroup $T(2)$ of the Wigner's little group, are connected with each-other for the specific relationships among the parameters of these transformation groups. In the language of BRST cohomology defined w.r.t. the conserved and nilpotent (co-)BRST charges, the (dual-)gauge transformed states turn out to be the sum of the original state and the (co-)BRST exact states. We comment on (i) the quasi-topological nature of the 4D free 2-form gauge theory from the degrees of freedom count on $e^{\mu\nu}(k)$, and (ii) the Wigner's little group and the BRST cohomology for the 2D one-form gauge theory *vis-à-vis* our analysis for the 4D 2-form gauge theory.

PACS numbers: 11 15-q; 02 40-k; 11 30-j

Keywords: (Dual-)gauge transformations; (co-)BRST symmetries; BRST cohomology; Hodge decomposition theorem; Wigner's little group

*E-mail address: malik@boson.bose.res.in

1 Introduction

It is a fairly well-established fact that the local gauge invariant theories govern the three (out of four) fundamental interactions of nature. In particular, the one-form (non-)Abelian gauge theories (which are at the heart of the standard model) have been verified and tested experimentally with stunning degree of precision. A couple of key features of these gauge theories are (i) the existence of the first-class constraints (in the language of Dirac's classification scheme [1,2]) on them that turn out to generate the local gauge transformations, and (ii) the interaction between gauge fields and matter fields is always dictated by the requirement of local gauge invariance, because of which, the gauge fields of the theory couple to the Noether conserved current (constructed by the matter fields). The latter are derived due to the presence of the continuous global gauge symmetry in the theory. For some of the free as well as interacting gauge theories, however, there exists a discrete symmetry transformation (for the Lagrangian density of these gauge theories) that corresponds to a certain specific kind of "duality" in the theory. This duality is found to be responsible for the existence of (i) a local dual-gauge symmetry transformation in the theory, and (ii) the analogue of the Hodge duality $*$ operation of differential geometry defined on a compact spacetime manifold without a boundary (see, e.g., [3-7]). Such (non-)interacting duality invariant gauge theories are found to provide a set of tractable field theoretical models for the Hodge theory. In this context, we can mention many interesting (gauge invariant) field theoretical models such as: (i) the free $(1+1)$-dimensional (2D) Abelian gauge theory [8-10], (ii) the interacting 2D Abelian gauge theory where there is an interaction between the $U(1)$ gauge field and the Dirac fields [11,12], (iii) the self-interacting 2D non-Abelian gauge theory where there is no interaction between matter fields and the non-Abelian gauge field [10,13], and (iv) the free Abelian 2-from gauge theory in $(3+1)$-dimensions (4D) [14].

For the covariant canonical quantization of any gauge theory (endowed with the first-class constraints), the Becchi-Rouet-Stora-Tyutin (BRST) formalism is one of the most elegant methods where the unitarity and "quantum" gauge (i.e. the BRST) invariance are respected together at any arbitrary order of perturbation theory. In this formalism, the usual local "classical" gauge transformations of the original gauge theory are traded with the "quantum" gauge transformations which are popularly known as the BRST transformations. One of the key features of these transformations is the nilpotency property. In fact, these symmetry transformations are generated by a conserved ($\dot{Q}_b = 0$) and nilpotent ($Q_b^2 = 0$) BRST charge Q_b. As it turns out, the first-class constraints of the original gauge theory are found to be encoded in the physicality condition $Q_b\,|phy>= 0$ in a subtle way. In fact, the condition $Q_b|phy>= 0$ implies that the physical states ($|phy>$) belong to the subspace of the total quantum Hilbert space of states (QHSS) that are annihilated by the BRST charge (or equivalently by the operator form of the first-class constraints of the original gauge theory). The nilpotency of the BRST charge and the physicality condition allow one to define the BRST cohomology where two quantum states ($|phy>'$ and $|phy>$)

are said to belong to the same cohomology class w.r.t. the BRST charge Q_b if they differ by a BRST exact state (i.e. $|phy>' = |phy> + Q_b\, |\Omega>$ for an arbitrary state $|\Omega>$ in the quantum Hilbert space). The BRST charge Q_b turns out to be the analogue of the exterior derivative d (with $d^2 = 0$) of differential geometry. Because of the presence of the duality, however, it is found that a local dual-gauge transformation also exists for these gauge theories. The latter transformation too can be traded with a nilpotent dual(co-)BRST symmetry transformations which are generated by a conserved ($\dot{Q}_d = 0$) and nilpotent ($Q_d^2 = 0$) co-BRST charge Q_d. Analogous to the cohomology w.r.t. Q_b, a co-cohomology can be defined w.r.t. the co-BRST charge Q_d. This charge turns out to correspond to the co-exterior derivative $\delta = \pm * d *$ (with $\delta^2 = 0$) of the differential geometry. As argued earlier, the discrete symmetry transformation of the theory (corresponding to the so-called "duality") turns out to be the analogue of the $*$ operation of the differential geometry (cf. (3.5) below). In particular, the discrete symmetry transformations on the ghost fields are exploited to define the anti-BRST and anti-co-BRST symmetries that are generated by the conserved and nilpotent ($Q_{ab}^2 = Q_{ad}^2 = 0$) anti-BRST (Q_{ab}) and anti-co-BRST (Q_{ad}) charges. Furthermore, the anti-commutator of the above nilpotent symmetries generates a bosonic symmetry transformation. The generator of this bosonic symmetry transformation Q_w turns out to be the analogue of the Laplacian operator Δ (i.e $\Delta = (d+\delta)^2 = \{d, \delta\}$) of the differential geometry. Thus, all the de Rham cohomological operators (d, δ, Δ) find their analogues in the language of some local, covariant and continuous symmetry transformations (and their generators) for the Lagrangian density of such a duality invariant gauge theory. Yet another symmetry in the theory is the ghost scale symmetry in which the (anti-)ghost fields transform by a scale transformation and the rest of the physical fields do not transform at all. This continuous symmetry transformation is generated by a conserved ghost charge Q_g. Thus, we have six local and conserved charges in the theory.

Having defined, described and discussed about the above conserved charges, now the stage is set for the statement of the celebrated Hodge decomposition theorem (HDT) in the QHSS. In fact, any arbitrary state $|\psi>_n$ with the ghost number n (i.e. $iQ_g|\psi>_n = n\ |\psi>_n$) in the QHSS can be expressed as the sum of a harmonic state $|\omega>_n$ (with $Q_w|\omega>_n = 0, Q_b|\omega>_n = 0, Q_d|\omega>_n = 0$), a BRST exact state $Q_b|\phi>_{n-1}$ and a BRST co-exact state $Q_d|\theta>_{n+1}$ as †

$$|\psi>_n = |\omega>_n + Q_b\ |\phi>_{n-1} + Q_d\ |\theta>_{n+1} \equiv |\omega>_n + Q_{ad}\ |\phi>_{n-1} + Q_{ab}\ |\theta>_{n+1}\ . \qquad (1.1)$$

Thus, it is crystal clear that there exists a *two-to-one mapping* between the conserved charges and the cohomological operators as: $(Q_b, Q_{ad}) \rightarrow d,\ (Q_d, Q_{ab}) \rightarrow \delta, \{Q_b, Q_d\} = \{Q_{ab}, Q_{ad}\} = Q_w \rightarrow \Delta$. All the above stated mathematical issues have been addressed and shown to be connected with the symmetry properties of the Lagrangian densities (and

†In the language of differential geometry defined on a compact manifold without a boundary, the HDT states that any arbitrary form f_n of degree n can be decomposed into a harmonic form h_n (with $\Delta h_n = 0, dh_n = 0, \delta h_n = 0$), an exact form de_{n-1} and a co-exact form δc_{n+1} as: $f_n = h_n + d\ e_{n-1} + \delta\ c_{n+1}$ where (d, δ, Δ) form the de Rham cohomological set of operators on the manifold.

their corresponding generators) for the field theoretical models of the gauge theories, we have stated earlier [8-14]. In a recent paper [15], the above cohomological properties and the quasi-topological nature of the 4D free Abelian 2-form gauge theory have been discussed by exploiting the (anti-)BRST and (anti-)co-BRST symmetries, their corresponding generators, their ensuing algebraic structure and the HDT in the QHSS.

A quite different but very interesting aspect of the above discussion is connected with the geometrical interpretations [16-21] for all the conserved charges ($Q_{(a)b}, Q_{(a)d}, Q_w$) and their two-to-one mappings with the cohomological operators (d, δ, Δ) in the framework of superfield formulation [22-26]. In fact, it has been demonstrated in our earlier works on the superfield formulation of the 2D (non-)Abelian gauge theories [16-21] that the conserved and nilpotent (anti-)BRST and (anti-)co-BRST charges correspond to the translational generators along the Grassmannian directions of the four (2+2)-dimensional supermanifold. In our formulation, it is interesting to state that (in spite of their common connection with the translational generators along the Grassmannian directions), one can make a clear distinction between $Q_{(a)b}$ and $Q_{(a)d}$ because of their radically different operations (and the corresponding ensuing effects) on the fermionic superfields which correspond to the anti-commuting (anti-)ghost fields of the theory (see, e.g., [21] for details). It has also been demonstrated that the 2D free Abelian and self-interacting non-Abelian gauge theories belong to a new class of topological field theories (TFTs) in flat spacetime (see, e.g., [3]) which capture together some of the key features of Witten and Schwarz type of TFTs [27-29]. It is gratifying to state that for these 2D TFTs, besides providing the geometrical origin for the conserved (and nilpotent) charges, the geometrical interpretation for the Lagrangian density and symmetric energy-momentum tensor has also been provided [19-21] in the language of translations of some local (but composite) superfields along the Grassmannian directions of the $(2 + 2)$-dimensional supermanifold. Mathematically, the above Lagrangian density and energy momentum tensor turn out to be the total derivatives w.r.t. the Grassmannian variables of the $(2 + 2)$-dimensional supermanifold.

As pointed out earlier, in general, it is the first-class constraints of the gauge theories that turn out to generate the "classical" local gauge transformations for the singular Lagrangian density of the theory. In the BRST formulation, these local symmetries of the gauge theory are traded with the nilpotent "quantum" gauge (i.e. BRST) symmetries. In a set of seminal papers [30-32] Weinberg first and later Han et al. [33-35] demonstrated the role of the translational subgroup $T(2)$ of the Wigner's little group in generating the local $U(1)$ gauge transformation for the massless (one-form) gauge field of the Maxwell theory. To be more precise, $T(2)$ (which is an Abelian invariant subgroup of the Wigner's little group) keeps the momentum k_μ of the light-like (massless) gauge particle invariant but it transforms the polarization vector $e_\mu(k)$ of the Maxwell one-form field in exactly the same manner as the $U(1)$ gauge transformation generated by the first-class constraints of the Maxwell theory. Thus, as far as the $U(1)$ gauge transformation for the one-form Maxwell field is concerned, there is an interesting complication between the translational symmetries associated with the $T(2)$ subgroup of the Wigner's (spacetime) little group and the internal symmetries related to the first-class constraints of the gauge theory.

The purpose of the present paper is to demonstrate that, for the $(3+1)$-dimensional (4D) free Abelian 2-form gauge theory (endowed with the (dual-)gauge transformations), the translation subgroup $T(2)$ of the Wigner's little group generates both the gauge [36] as well as the dual-gauge transformations on the anti-symmetric polarization tensor $e^{\mu\nu}(k)$ of the 2-form gauge field $B_{\mu\nu}(x)$ for the specific choices of the (dual-)gauge transformation parameters in terms of the parameters of the $T(2)$ subgroup (cf. (5.12), (5.16), (5.23), (5.24) below). This result is also discussed in the framework of BRST formalism. In our expositions: (i) the HDT in the QHSS, (ii) the BRST cohomology, and (iii) the (co-)BRST symmetries play very decisive roles. For instance, first of all, the physical state ($|phy>$) (as well as the physical vacuum) of the theory is chosen to be the harmonic state of the Hodge decomposed state of any arbitrary state in the QHSS. This immediately implies that $Q_{b(ad)}|phy>=0, Q_{d(ab)}|phy>=0, Q_w|phy>=0$. As a consequence of the above requirements, we obtain certain specific type of restrictions on the single particle quantum state (SPQS) of the 2-form gauge field $B_{\mu\nu}(x)$. These constraints, in turn, provide very informative relationships between the anti-symmetric polarization tensor $e^{\mu\nu}(k)$ and the momentum 4-vector k^μ which are found to be responsible for *the first step* of reduction of the number of degrees of freedom associated with $e^{\mu\nu}(k)$ in 4D. At *the next step*, it is the nilpotent (co-)BRST (or (dual-)gauge) symmetry transformations that dictate the reduction process of the degrees of freedom of $e^{\mu\nu}(k)$. At this stage, (i) we demonstrate the connection between the $T(2)$ subgroup of the Wigner's little group and the (dual-)gauge (or (co-)BRST) symmetry transformation group when they operate on the *doubly reduced* polarization tensor $e^{\mu\nu}(k)$, and (ii) we comment on the quasi-topological nature [15] of the 4D 2-form Abelian gauge theory in the framework of an extended BRST formalism. Ultimately, in the language of the BRST cohomology w.r.t. the (co-)BRST charges, the (dual-)gauge (or (co-)BRST) transformed states (which are connected with the transformation generated by $T(2)$ subgroup of the Wigner's little group on the polarization tensor $e^{\mu\nu}(k)$) turn out to be the sum of the original SPQS plus a BRST (co-)exact state. For the 2D free Abelian one-form gauge theory, it is not possible to apply the key concepts of the Wigner's little group and its connection to the (dual-)gauge transformations. This is because of the fact that one can gauge away both the components of the 2D polarization vector by the choice of the (dual-)gauge parameters of the (dual-)gauge transformations. Thus, nothing remains in the theory and this theory becomes topological in nature [8-10,19]. This fact is reflected in the matrix representation of the Wigner's little group which becomes identity (trivial) for the free Abelian one-form gauge theory in 2D. As a consequence, neither momentum nor polarization vectors transform under the Wigner's little group defined for the 2D Abelian one-form gauge theory. However, in the language of constraints, BRST cohomology and HDT, one can capture mathematically as well as physically the key points of the (dual-)gauge transformations in an elegant way (see, e.g., section 3 below).

The material of our work is organized as follows. In section 2, we define, discuss and distinguish between the gauge and dual-gauge symmetry transformations for the gauge-fixed version of the Lagrangian densities of the 2D free Abelian one-form and 4D free Abelian 2-form gauge theories. Section 3 is devoted to the discussion of (co-)BRST symmetries. We obtain the normal mode expansion for the basic fields of both the theories and discuss about the BRST cohomology and physicality condition in section 4. The central of our present paper is section 5 where we establish the connection between $T(2)$ subgroup of the Wigner's little group and the (dual-)gauge transformations. Furthermore, we express this connection in the language of BRST cohomology and comment on the quasi-topological nature [15] of the 4D 2-form gauge theory. Finally, in section 6, we make some concluding remarks and point out a few future directions.

2 (Dual-)gauge transformations: Lagrangian formulation

Let us begin with the gauge-fixed Lagrangian density $\mathcal{L}_0^{(1)}$ for a two $(1+1)$-dimensional free Abelian gauge theory in the Feynman gauge ‡ (see, e.g., [37-40])

$$\mathcal{L}_0^{(1)} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}(\partial\cdot A)^2 \equiv \tfrac{1}{2}E^2 - \tfrac{1}{2}(\partial\cdot A)^2, \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the anti-symmetric second-rank curvature tensor (with $F_{01} = E$ = electric field) defined through the 2-form $F = dA = \frac{1}{2}(dx^\mu \wedge dx^\nu)(F_{\mu\nu})$. As is evident, this 2-form is derived by the application of the exterior derivative $d = dx^\mu\partial_\mu$ (with $d^2 = 0$) on the connection one-form $A = dx^\mu A_\mu$ (where A_μ is the vector potential) and the gauge-fixing term $(\partial\cdot A) = (-*d*A)$ is defined through the application of the dual-exterior derivative $\delta = -*d*$ (with $\delta^2 = 0$) on the one-form A. Here, for the 2D theory, the curvature tensor $F_{\mu\nu}$ has no magnetic component and the $*$ operation is the Hodge duality operation on 2D spacetime manifold. The application of the Laplacian operator $\Delta = (d+\delta)^2 = d\delta + \delta d$ on the one-form A leads to $\Delta A = dx^\mu \Box A_\mu$. In fact, the equation of motion ($\Box A_\mu = 0$), emerging from the above gauge-fixed Lagrangian density, is captured by the Laplacian operator in the sense that it (i.e. $\Box A_\mu = 0$) can be derived from the requirement of the validity of the Laplace equation $\Delta A = 0$. Together the set of geometrical operators (d, δ, Δ) define the de Rham cohomological properties of the differential forms on a compact manifold without a boundary and obey the algebra: $d^2 = \delta^2 = 0, \Delta = (d+\delta)^2 = \{d, \delta\}, [\Delta, d] = [\Delta, \delta] = 0$.

It is straightforward to check that the above Lagrangian density, under the following local $U(1)$ gauge and dual-gauge transformations

$$\begin{array}{lcl} A_\mu(x) & \rightarrow & A_\mu^{(g)}(x) = A_\mu(x) + \partial_\mu\alpha(x), \\ A_\mu(x) & \rightarrow & A_\mu^{(dg)}(x) = A_\mu(x) - \varepsilon_{\mu\nu}\partial^\nu\Sigma(x), \end{array} \tag{2.2}$$

‡We adopt here the conventions and notations such that the two dimensional flat Minkowski metric $\eta_{\mu\nu} = \text{diag}\,(+1, -1)$ and the totally anti-symmetric Levi-Civita tensor $\varepsilon_{\mu\nu} = -\varepsilon^{\mu\nu}, \varepsilon_{01} = +1 = \varepsilon^{10}, (\partial\cdot A) = \partial_0 A_0 - \partial_1 A_1, E = -\varepsilon^{\mu\nu}\partial_\mu A_\nu = \partial_0 A_1 - \partial_1 A_0 = F_{01}, \Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = (\partial_0)^2 - (\partial_1)^2$. Here the Greek indices $\mu, \nu, \kappa.... = 0, 1$ stand for the spacetime directions on the 2D manifold.

remains invariant if the local infinitesimal transformation parameters $\alpha(x)$ and $\Sigma(x)$ are restricted to obey $\Box\alpha(x) = \Box\Sigma(x) = 0$. At this stage, some of the key and relevant points are (i) under the gauge and dual-gauge transformations, it is the kinetic energy term (more precisely the electric field E itself) and the gauge-fixing term (more accurately $(\partial \cdot A)$ itself) remain invariant (i.e. $E \to E^{(g)} = E, (\partial \cdot A) \to (\partial \cdot A)^{(dg)} = (\partial \cdot A)$), respectively. (ii) Exactly identical restrictions (i.e. $\Box\alpha = \Box\Sigma = 0$) on the transformation parameters emerge for the invariance of the gauge-fixing term under the gauge transformation (i.e. $(\partial \cdot A) \to (\partial \cdot A)^{(g)} = (\partial \cdot A)$ for $\Box\alpha = 0$) and the invariance of the curvature term (i.e. the electric field) under the dual-gauge transformation (i.e. $E \to E^{(dg)} = E$ for $\Box\Sigma = 0$). (iii) The latter transformation in (2.2) are christened as the dual gauge transformation because $(\partial \cdot A)$ and E are 'Hodge dual' to each-other from the point of view of their derivation by the application of cohomological operators δ and d on the connection one-form A. (iv) It is interesting to note that, under a couple of independent discrete symmetry transformations

$$\partial_\mu \to \pm i\varepsilon_{\mu\nu}\partial^\nu, \qquad A_\mu(x) \to A_\mu(x), \tag{2.3a}$$

$$A_\mu(x) \to \mp i\varepsilon_{\mu\nu}A^\nu(x), \qquad \partial_\mu \to \partial_\mu, \tag{2.3b}$$

the Lagrangian density (2.1) remains invariant because the kinetic energy and the gauge-fixing terms exchange with each-other. Mathematically, this statement can be succinctly expressed as

$$(\partial \cdot A) \to \pm iE, \qquad E \to \pm i(\partial \cdot A), \qquad \mathcal{L}_0^{(1)} \to \mathcal{L}_0^{(1)}. \tag{2.4}$$

A proper generalization of equations (2.3) will turn out to be the analogue of the $*$ operation of differential geometry as we shall see later in the framework of BRST formalism.

Equipped with our understanding of the 2D free Abelian gauge theory which sets the backdrop, let us dwell a bit on the 4D free Abelian 2-form gauge theory described by the gauge-fixed Lagrangian density in the Feynman gauge § (see, e.g., [40,14,15])

$$\mathcal{L}_0^{(2)} = \tfrac{1}{12} H^{\mu\nu\kappa} H_{\mu\nu\kappa} + \tfrac{1}{2}(\partial_\mu B^{\mu\nu})(\partial^\kappa B_{\kappa\nu}), \tag{2.5}$$

where the totally anti-symmetric curvature tensor $H_{\mu\nu\kappa} = \partial_\mu B_{\nu\kappa} + \partial_\nu B_{\kappa\mu} + \partial_\kappa B_{\mu\nu}$ is derived from the 3-form $H = dB = \frac{1}{3!}(dx^\mu \wedge dx^\nu \wedge dx^\kappa)(H_{\mu\nu\kappa})$ by application of the exterior derivative d on the connection 2-form $B = \frac{1}{2}(dx^\mu \wedge dx^\nu)(B_{\mu\nu})$. The application of the dual exterior derivative $\delta = - * d *$ on the 2-form B leads to the definition of the one-form gauge-fixing term $(\partial^\kappa B_{\kappa\mu})(dx^\mu) = \delta B$. The action of the Laplacian operator Δ on the 2-form basic field B yields $\Delta B = \frac{1}{2}(dx^\mu \wedge dx^\nu)(\Box B_{\mu\nu})$ where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = (\partial_0)^2 - (\partial_i)^2$. Thus, the equation of motion $\Box B_{\mu\nu} = 0$ for the above gauge-fixed Lagrangian density is contained

§We follow here the conventions and notations in such a way that the flat 4D Minkowski spacetime manifold is endowed with the flat metric $\eta_{\mu\nu} = \text{diag}\,(+1, -1, -1, -1)$ and the totally antisymmetric Levi-Civita tensor obeys $\varepsilon_{\mu\nu\kappa\sigma}\varepsilon^{\mu\nu\kappa\sigma} = -4!$, $\varepsilon_{\mu\nu\kappa\sigma}\varepsilon^{\mu\nu\kappa\eta} = -3!\delta^\eta_\sigma, etc.$, $\varepsilon_{0123} = +1 = -\varepsilon^{0123}$, $\varepsilon_{0ijk} = \varepsilon_{ijk}$. Here the Greek indices $\mu, \nu, \kappa ..... = 0, 1, 2, 3$ correspond to the spacetime directions on the 4D manifold and the Latin indices $i, j, k, ... = 1, 2, 3$ stand only for the space directions of the same manifold.

in the requirement that the Laplace equation $\Delta B = 0$ is satisfied. The above Lagrangian density remains invariant under the following gauge and dual-gauge transformations

$$\begin{array}{lll} B_{\mu\nu} & \rightarrow & B^{(g)}_{\mu\nu} = B_{\mu\nu} + (\partial_\mu \alpha_\nu - \partial_\nu \alpha_\mu), \\ B_{\mu\nu} & \rightarrow & B^{(dg)}_{\mu\nu} = B_{\mu\nu} + \varepsilon_{\mu\nu\kappa\xi} \partial^\kappa \Sigma^\xi, \end{array} \tag{2.6}$$

if the local infinitesimal parameters of the above transformations are restricted to satisfy $\Box \alpha_\mu - \partial_\mu (\partial \cdot \alpha) = 0,\ \Box \Sigma_\mu - \partial_\mu (\partial \cdot \Sigma) = 0$. Some of the salient features, at this juncture, are (i) it is the curvature term $H_{\mu\nu\kappa}$ (derived from $H = dB$) and the gauge-fixing term $\partial^\mu B_{\mu\nu}$ (derived from $\delta B = \partial^\kappa B_{\kappa\mu} dx^\mu$) that are found to remain invariant under the gauge- and dual-gauge transformations, respectively. (ii) The restrictions ($\Box \alpha_\mu - \partial_\mu (\partial \cdot \alpha) = 0,\ \Box \Sigma_\mu - \partial_\mu (\partial \cdot \Sigma) = 0$) on the local infinitesimal transformation parameters $\alpha_\mu(x)$ and $\Sigma_\mu(x)$ are identical for both the above transformations. (iii) The above restrictions on the transformation parameters α_μ and Σ_μ emerge when we demand (a) the invariance of the gauge-fixing term under the gauge transformation (i.e. $(\partial_\mu B^{\mu\nu}) \rightarrow (\partial_\mu B^{\mu\nu})^{(g)} = (\partial_\mu B^{\mu\nu})$), and (b) the invariance of the kinetic energy term under the dual-gauge transformation (i.e. $H_{\mu\nu\kappa} H^{\mu\nu\kappa} \rightarrow (H_{\mu\nu\kappa} H^{\mu\nu\kappa})^{(dg)} = H_{\mu\nu\kappa} H^{\mu\nu\kappa}$). To be more elaborate on this latter point, let us consider the explicit infinitesimal version (δ_D) of the dual-gauge transformation (2.6) applied to the variation of the Lagrangian density

$$\delta_D \mathcal{L}^{(2)}_0 = \tfrac{1}{6} H^{\mu\nu\kappa} \delta_D (H_{\mu\nu\kappa}) = \tfrac{1}{2} H^{\mu\nu\kappa} \varepsilon_{\nu\kappa\sigma\xi} \partial_\mu \partial^\sigma \Sigma^\xi, \tag{2.7}$$

where $\delta_D B_{\mu\nu} = \varepsilon_{\mu\nu\kappa\zeta} \partial^\kappa \Sigma^\zeta,\ \delta_D (\partial_\mu B^{\mu\nu}) = 0$. Ultimately, the expansion of the above term in explicit components, implies the following

$$\begin{array}{l} \tfrac{1}{2} H^{\mu\nu\kappa} \varepsilon_{\nu\kappa\sigma\xi} \partial_\mu \partial^\sigma \Sigma^\xi = H^{\mu\nu\kappa} \varepsilon_{\mu\nu\kappa\xi} \left[\, \Box \Sigma^\xi - \partial^\xi (\partial \cdot \Sigma) \,\right], \\ \delta_D \mathcal{L}^{(2)}_0 = 0 \quad \rightarrow \quad \Box \Sigma_\mu - \partial_\mu (\partial \cdot \Sigma) = 0. \end{array} \tag{2.8}$$

It is easy to obtain the above condition by choosing $H_{\mu\nu\kappa} = \varepsilon_{\mu\nu\kappa\xi} V^\xi$ and showing that

$$\begin{array}{l} \delta_D \mathcal{L}^{(2)}_0 = 0 \quad \rightarrow \quad V^\mu [\Box \Sigma_\mu - \partial_\mu (\partial \cdot \Sigma)] = 0, \\ (\tfrac{1}{12} H^{\mu\nu\kappa} H_{\mu\nu\kappa}) \quad \rightarrow \quad (\tfrac{1}{12} H^{\mu\nu\kappa} H_{\mu\nu\kappa})^{(dg)} = (\tfrac{1}{12} H^{\mu\nu\kappa} H_{\mu\nu\kappa}). \end{array} \tag{2.9}$$

Thus, the condition (2.8) emerges very naturally for the non-zero axial-vector V_μ. (iv) The above gauge-fixed Lagrangian density remains invariant (i.e. $\mathcal{L}^{(2)}_0 \rightarrow \mathcal{L}^{(2)}_0$) under the following discrete symmetry transformation

$$B_{\mu\nu} \rightarrow \mp \tfrac{i}{2} \varepsilon_{\mu\nu\kappa\xi} B^{\kappa\xi}, \tag{2.10}$$

because the kinetic energy and gauge-fixing terms of the above Lagrangian density exchange with each-other (i.e. $\frac{1}{12} H^{\mu\nu\kappa} H_{\mu\nu\kappa} \leftrightarrow \frac{1}{2} (\partial_\mu B^{\mu\nu})(\partial^\kappa B_{\kappa\nu})$) under (2.10).

The gauge-fixed Lagrangian density (2.5) can be generalized by introducing a massless ($\Box \phi_1 = 0$) scalar field ϕ_1 in the gauge-fixing term, as

$$\mathcal{L}^{(2)}_1 = \tfrac{1}{12} H^{\mu\nu\kappa} H_{\mu\nu\kappa} + \tfrac{1}{2} (\partial_\mu B^{\mu\nu} - \partial^\nu \phi_1)(\partial^\kappa B_{\kappa\nu} - \partial_\nu \phi_1). \tag{2.11}$$

It is interesting to note that (i) the equation of motion ($\Box B_{\mu\nu} = 0$) for the 2-form gauge field $B_{\mu\nu}$ remains intact in spite of the presence of the massless scalar field ϕ_1. (ii) The scalar field ϕ_1 does not transform under the (dual-)gauge transformations discussed above. (iii) The Euler-Lagrange equation of motion for ϕ_1 field is $\Box\phi_1 = 0$. (iv) The kinetic energy term of the Lagrangian density (2.14) can be generalized to include another massless ($\Box\phi_2 = 0$) scalar field ϕ_2, as given below

$$\begin{aligned} \mathcal{L}_2^{(2)} &= \tfrac{1}{2}(\partial_\mu B^{\mu\nu} - \partial^\nu \phi_1)(\partial^\kappa B_{\kappa\nu} - \partial_\nu \phi_1) \\ &- \tfrac{1}{2}(\tfrac{1}{2}\varepsilon_{\mu\nu\kappa\zeta}\partial^\nu B^{\kappa\zeta} - \partial_\mu \phi_2)(\tfrac{1}{2}\varepsilon^{\mu\sigma\eta\xi}\partial_\sigma B_{\eta\xi} - \partial^\mu \phi_2). \end{aligned} \tag{2.12}$$

The above Lagrangian density (2.12) leads to the following equations of motion

$$\Box B_{\mu\nu} = 0, \qquad \Box\phi_1 = 0, \qquad \Box\phi_2 = 0. \tag{2.13}$$

Under the following generalization of the discrete symmetry transformations (2.10)

$$B_{\mu\nu} \to \mp \tfrac{i}{2}\varepsilon_{\mu\nu\kappa\xi} B^{\kappa\xi}, \qquad \phi_1 \to \pm i\phi_2, \qquad \phi_2 \to \mp i\phi_1, \tag{2.14}$$

the Lagrangian density (2.12) remains invariant (i.e. $\mathcal{L}_2^{(2)} \to \mathcal{L}_2^{(2)}$).

3 (Co-)BRST symmetries: on-shell nilpotent versions

The (dual-)gauge transformations for the gauge-fixed Lagrangian densities of the 2D one-form and 4D 2-form gauge theories can be traded with the (co-)BRST symmetries which turn out to be (the off-shell as well as on-shell) nilpotent of order two. The BRST invariant version of the Lagrangian density (2.1) is (see, e.g., [8-10])

$$\mathcal{L}_b^{(1)} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}(\partial\cdot A)^2 - i\partial_\mu \bar{C}\partial^\mu C \equiv \tfrac{1}{2}E^2 - \tfrac{1}{2}(\partial\cdot A)^2 - i\partial_\mu \bar{C}\partial^\mu C, \tag{3.1}$$

where the (anti-)ghost fields $(\bar{C})C$ are anti-commuting ($\bar{C}C + C\bar{C} = 0, C^2 = \bar{C}^2 = 0$) in nature. It is straightforward to check that (3.1) remains invariant under the following on-shell ($\Box C = \Box\bar{C} = 0$) nilpotent ($s_{ab}^2 = s_b^2 = 0$) (anti-)BRST $s_{(a)b}$ transformations ¶ on the basic fields of the theory (with $s_{ab}s_b + s_b s_{ab} = 0$)

$$\begin{aligned} &s_b A_\mu = \partial_\mu C, \qquad s_b C = 0, \qquad s_b \bar{C} = -i(\partial\cdot A), \\ &s_{ab} A_\mu = \partial_\mu \bar{C}, \qquad s_{ab}\bar{C} = 0, \qquad s_{ab} C = +i(\partial\cdot A). \end{aligned} \tag{3.2}$$

The salient points, at this stage, are (i) the Lagrangian density transforms to a total derivative under (3.2). (ii) There are no restrictions on the (anti-)ghost fields $(\bar{C})C$. (iii) The gauge transformation parameter α of (2.2) has been replaced by an anti-commuting number χ and the ghost field C. (iv) The restriction $\Box\alpha = 0$ on the local infinitesimal

¶We follow here the notations and conventions adopted by Weinberg [37]. In fact, in its totality, the (anti-)BRST transformations $\delta_{(A)B}$ (with $\delta_{(A)B}^2 = 0$) are product ($\delta_{(A)B} = \chi s_{(a)b}$) of an anti-commuting (i.e. $\chi C + C\chi = 0$, $\chi\bar{C} + \bar{C}\chi = 0$, etc.) spacetime independent parameter χ and $s_{(a)b}$ with $s_{(a)b}^2 = 0$.

parameter α of the gauge transformation (2.2) is not required here because the equation of motion $\Box C = 0$ takes care of it. (v) There are two nilpotent ($s_{(a)b}^2 = 0$) (anti-)BRST transformations (3.2) corresponding to a single gauge transformation in (2.2).

It is interesting to note that the dual-gauge transformation of the gauge field $A_\mu(x)$ (cf. (2.2)) can be generalized to the dual(co-)BRST symmetry transformation by the replacement $\Sigma(x) = \chi \bar{C}(x)$. The ensuing on-shell ($\Box C = \Box \bar{C} = 0$) nilpotent $s_{(a)d}^2 = 0$ (anti-)co-BRST transformations ($s_{(a)d}$) (with $s_{ad} s_d + s_d s_{ad} = 0$)

$$\begin{array}{lll} s_d A_\mu = -\varepsilon_{\mu\nu} \partial^\nu \bar{C}(x), & s_d \bar{C} = 0, & s_d C = -iE, \\ s_{ad} A_\mu = -\varepsilon_{\mu\nu} \partial^\nu C, & s_{ad} C = 0, & s_{ad} \bar{C} = +iE, \end{array} \tag{3.3}$$

leave the Lagrangian density (3.1) invariant up to a total derivative. The (anti-)ghost fields $(\bar{C})C$ are restricted in *no way* for the consideration of the co-BRST symmetry transformations unlike the local transformation parameter $\Sigma(x)$ of the dual-gauge transformation. It is obvious that a couple of on-shell ($\Box C = \Box \bar{C} = 0$) nilpotent ($s_{(a)d}^2 = 0$) (anti-)co-BRST symmetry transformations emerge from a single dual-gauge transformation of (2.2). Furthermore, the Lagrangian density (3.1) remains invariant under the following two independent discrete transformations

$$\partial_\mu \to \pm i \varepsilon_{\mu\nu} \partial^\nu, \qquad A_\mu(x) \to A_\mu(x), \qquad C(x) \to \pm i \bar{C}(x), \qquad \bar{C}(x) \to \pm i C(x), \tag{3.4a}$$

$$A_\mu(x) \to \mp i \varepsilon_{\mu\nu} A^\nu(x), \qquad \partial_\mu \to \partial_\mu, \qquad C(x) \to \pm i \bar{C}(x), \qquad \bar{C}(x) \to \pm i C(x), \tag{3.4b}$$

which are nothing but the generalization of transformations (2.3) to include the transformations on the (anti-)ghost fields. Now the stage is set to provide the meaning for the Hodge duality $*$ operation of differential geometry in the language of BRST type symmetries. It can be checked clearly that, for any generic field $\Phi = A_\mu, C, \bar{C}$ of the theory, the following relationship (see, e.g., [10])

$$s_{(a)d} \Phi = \pm \; * \; s_{(a)b} \; * \; \Phi, \tag{3.5}$$

is valid. Here the Hodge duality $*$ operation in the above equation corresponds to the discrete symmetry transformations (3.4). The $(+)-$ signs in the above equation are dictated by the similar signs that appear when two successive Hodge duality $*$ operations are applied on the generic field $\Phi = A_\mu, C, \bar{C}$ as expressed below [41,10]

$$* \; (\; * \;) \Phi = \pm \; \Phi. \tag{3.6}$$

It will be noted that the signs on the r.h.s. of the above equation, in general, may differ for the discrete transformations in (3.4a) and (3.4b) (see, e.g., [10]). It is obvious that the relationship in (3.5) is reminiscent of the connection between the (dual-)exterior derivatives $(\delta)d$ which are related to each-other (i.e. $\delta = \pm * d *$). In the realm of differential geometry, the $(+)-$ signs are dictated by the dimensionality of the manifold on which d, δ and the Hodge duality $*$ operation are defined (see, e.g., [3]).

The off-shell nilpotent (anti-)BRST invariant version of the Lagrangian density (2.11) for the 4D free 2-form Abelian gauge theory that includes bosonic (anti-)ghost fields $(\bar{\beta})\beta$

(with $\beta^2 \neq 0, \bar{\beta}^2 \neq 0$) as well as the fermionic vector (anti-)ghost fields $(\bar{C}_\mu)C_\mu$ (with $C_\mu C_\nu + C_\nu C_\mu = 0, C_\mu \bar{C}_\nu + \bar{C}_\nu C_\mu = 0, \bar{C}_\mu \bar{C}_\nu + \bar{C}_\nu \bar{C}_\mu = 0, (C_\mu)^2 = 0, (\bar{C}_\mu)^2 = 0$ etc.) is [14]

$$\begin{aligned} \mathcal{L}_{b1}^{(2)} &= \tfrac{1}{12} H^{\mu\nu\kappa} H_{\mu\nu\kappa} + \tfrac{1}{2} B^\mu\, (\partial^\kappa B_{\kappa\mu} - \partial_\mu \phi_1) - \tfrac{1}{2} B^\mu B_\mu - \partial_\mu \bar{\beta} \partial^\mu \beta \\ &+ (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu)\, (\partial^\mu C^\nu) + \rho\, (\partial \cdot C + \lambda) + (\partial \cdot \bar{C} + \rho)\, \lambda, \end{aligned} \tag{3.7}$$

where B_μ is the bosonic auxiliary field and $(\rho)\lambda$ are the fermionic ($\rho^2 = \lambda^2 = 0, \rho\lambda + \lambda\rho = 0$) scalar (anti-)ghost auxiliary fields. In fact, one can linearize the quadratic kinetic energy term ($\frac{1}{12} H^{\mu\nu\kappa} H_{\mu\nu\kappa}$) by introducing a couple of bosonic auxiliary fields $\mathcal{B}_\mu$ and ϕ_2 as [14,15]

$$\begin{aligned} \mathcal{L}_{b2}^{(2)} &= \tfrac{1}{2} \mathcal{B}^\mu \mathcal{B}_\mu - \mathcal{B}^\mu\, (\tfrac{1}{2} \varepsilon_{\mu\nu\kappa\zeta} \partial^\nu B^{\kappa\zeta} - \partial_\mu \phi_2) + \tfrac{1}{2} B^\mu\, (\partial^\kappa B_{\kappa\mu} - \partial_\mu \phi_1) - \tfrac{1}{2} B^\mu B_\mu \\ &- \partial_\mu \bar{\beta} \partial^\mu \beta + (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu)\, (\partial^\mu C^\nu) + \rho\, (\partial \cdot C + \lambda) + (\partial \cdot \bar{C} + \rho)\, \lambda. \end{aligned} \tag{3.8}$$

The above Lagrangian density respects both the off-shell nilpotent ($s_{(D)B}^2 = 0$) (co-)BRST ($s_{(D)B}$) symmetry transformations as listed below [14,15]

$$\begin{aligned} s_B B_{\mu\nu} &= (\partial_\mu C_\nu - \partial_\nu C_\mu), \qquad s_B C_\mu = \partial_\mu \beta, \qquad s_B \bar{C}_\mu = B_\mu, \\ s_B \phi_1 &= -\lambda, \qquad s_B \bar{\beta} = \rho, \qquad s_B(\beta, \mathcal{B}_\mu, \lambda, \rho, B_\mu, \phi_2, H_{\mu\nu\kappa}) = 0, \end{aligned} \tag{3.9}$$

$$\begin{aligned} s_D B_{\mu\nu} &= \varepsilon_{\mu\nu\kappa\zeta} \partial^\kappa \bar{C}^\zeta, \quad s_D \bar{C}_\mu = -\partial_\mu \bar{\beta}, \quad s_D C_\mu = \mathcal{B}_\mu, \quad s_D \beta = \lambda, \\ s_D \phi_2 &= -\rho, \qquad s_D(\bar{\beta}, \mathcal{B}_\mu, \lambda, \rho, B_\mu, \phi_1, \partial^\mu B_{\mu\kappa}) = 0. \end{aligned} \tag{3.10}$$

It is interesting to point out that the off-shell nilpotent BRST symmetries (3.9) are also respected by the Lagrangian density (3.7). If we substitute the following equations of motion emerging from the Lagrangian density (3.8)

$$\begin{aligned} B_\mu &= (\partial^\nu B_{\nu\mu} - \partial_\mu \phi_1), \qquad \rho = -\tfrac{1}{2}\, (\partial \cdot \bar{C}), \\ \mathcal{B}_\mu &= (\tfrac{1}{2} \varepsilon_{\mu\nu\kappa\zeta} \partial^\nu B^{\kappa\zeta} - \partial_\mu \phi_2), \qquad \lambda = -\tfrac{1}{2}\, (\partial \cdot C), \end{aligned} \tag{3.11}$$

into (3.8) itself, we obtain the following Lagrangian density

$$\begin{aligned} \mathcal{L}_{b3}^{(2)} &= \tfrac{1}{2} (\partial^\kappa B_{\kappa\mu} - \partial_\mu \phi_1)(\partial_\nu B^{\nu\mu} - \partial^\mu \phi_1) \\ &- \tfrac{1}{2} (\tfrac{1}{2} \varepsilon_{\mu\nu\kappa\zeta} \partial^\nu B^{\kappa\zeta} - \partial_\mu \phi_2)(\tfrac{1}{2} \varepsilon^{\mu\xi\sigma\eta} \partial_\xi B_{\sigma\eta} - \partial^\mu \phi_2) \\ &- \partial_\mu \bar{\beta} \partial^\mu \beta + (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu)\, (\partial^\mu C^\nu) - \tfrac{1}{2}\, (\partial \cdot \bar{C})(\partial \cdot C), \end{aligned} \tag{3.12}$$

which turns out to respect the on-shell ($\Box B_{\mu\nu} = \Box \phi_1 = \Box \phi_2 = \Box \beta = \Box \bar{\beta} = 0, \Box C_\mu = \frac{3}{2} \partial_\mu (\partial \cdot C), \Box \bar{C}_\mu = \frac{3}{2} \partial_\mu (\partial \cdot \bar{C})$) nilpotent ($\tilde{s}_{(d)b}^2 = 0$) version of the (co-)BRST symmetry transformations as expressed below

$$\begin{aligned} \tilde{s}_b B_{\mu\nu} &= (\partial_\mu C_\nu - \partial_\nu C_\mu), \qquad \tilde{s}_b C_\mu = \partial_\mu \beta, \qquad \tilde{s}_b \bar{C}_\mu = (\partial^\nu B_{\nu\mu} - \partial_\mu \phi_1), \\ \tilde{s}_b \phi_1 &= +\tfrac{1}{2} (\partial \cdot C), \qquad \tilde{s}_b \bar{\beta} = -\tfrac{1}{2} (\partial \cdot \bar{C}), \qquad \tilde{s}_b(\beta, \phi_2, H_{\mu\nu\kappa}) = 0, \end{aligned} \tag{3.13}$$

$$\begin{aligned} \tilde{s}_d B_{\mu\nu} &= \varepsilon_{\mu\nu\kappa\zeta} \partial^\kappa \bar{C}^\zeta, \quad \tilde{s}_d \bar{C}_\mu = -\partial_\mu \bar{\beta}, \quad \tilde{s}_d C_\mu = (\tfrac{1}{2} \varepsilon_{\mu\nu\kappa\sigma} \partial^\nu B^{\kappa\sigma} - \partial_\mu \phi_2), \\ \tilde{s}_d \phi_2 &= +\tfrac{1}{2} (\partial \cdot \bar{C}), \qquad \tilde{s}_d \beta = -\tfrac{1}{2} (\partial \cdot C), \quad \tilde{s}_d(\bar{\beta}, \phi_1, \partial^\mu B_{\mu\kappa}) = 0. \end{aligned} \tag{3.14}$$

There are specific pertinent points in order here. First, it will be noted that the Lagrangian density (3.12) is the generalized version of (2.12) which respects both the on-shell nilpotent (co-)BRST symmetries. Second, the generalization of the discrete symmetry transformations (2.14) for the Lagrangian density (3.8) (that respects the off-shell nilpotent (co-)BRST symmetry transformations (3.9) and (3.10)) is as follows [14,15]

$$
\begin{array}{lll}
B_{\mu\nu} & \rightarrow & \mp\frac{i}{2}\varepsilon_{\mu\nu\kappa\xi}B^{\kappa\xi}, \qquad \phi_1 \rightarrow \pm i\phi_2, \qquad \phi_2 \rightarrow \mp i\phi_1, \\
B_\mu & \rightarrow & \pm i\mathcal{B}_\mu, \quad \mathcal{B}_\mu \rightarrow \mp iB_\mu, \quad \beta \rightarrow \mp i\bar{\beta}, \quad \bar{\beta} \rightarrow \pm i\beta, \\
C_\mu & \rightarrow & \pm i\bar{C}_\mu, \quad \bar{C}_\mu \rightarrow \pm iC_\mu, \quad \rho \rightarrow \pm i\lambda, \quad \lambda \rightarrow \pm i\rho.
\end{array}
\tag{3.15}
$$

The on-shell version of the above discrete symmetry transformations for the Lagrangian density (3.12) can be obtained from the substitution of the equations of motion (3.11) for the auxiliary fields into (3.15). Third, the on-shell as well as off-shell version of the anti-BRST and anti-co-BRST transformations can be obtained from (3.9), (3.10), (3.13) and (3.14) by exploiting the discrete transformations (3.15) for the bosonic as well as fermionic (anti-)ghost fields. Finally, all the symmetry transformations of the theory can be generically expressed in terms of the symmetry generators Q_r as (see, e.g., [10,14])

$$
s_r\Phi = -i\ [\Phi, Q_r]_\pm, \qquad r = b, ab, d, ad, g, w,
\tag{3.16}
$$

where the subscripts $(+)-$ on the brackets correspond to (anti-)commutators for the generic field Φ being (fermionic)bosonic in nature and g, w correspond to the existence of a ghost symmetry and a bosonic symmetry. The above generic expression is valid for the 2D as well as 4D theories for the off-shell as well as on-shell nilpotent version of the (anti-) BRST and (anti-)co-BRST symmetries together with other symmetries of the theories [10,14].

4 BRST cohomology: physical state condition

Here we shall recall some of the key and pertinent points of earlier work (see, e.g., [9]) on the Hodge decomposition theorem for the 2D free Abelian gauge theory. To this end in mind, we first express the normal mode expansion for the basic fields $(A_\mu, C, \bar{C})$ of the Lagrangian density (3.1) in the (momentum) phase space as (see, e.g.,[9,37])

$$
\begin{array}{lll}
A_\mu(x) & = & \displaystyle\int \frac{dk}{(2\pi)^{1/2}(2k^0)^{1/2}}\ [\ a^\dagger_\mu(k)e^{ik\cdot x} + a_\mu(k)e^{-ik\cdot x}\ ], \\
C(x) & = & \displaystyle\int \frac{dk}{(2\pi)^{1/2}(2k^0)^{1/2}}\ [\ c^\dagger(k)e^{ik\cdot x} + c(k)e^{-ik\cdot x}\ ], \\
\bar{C}(x) & = & \displaystyle\int \frac{dk}{(2\pi)^{1/2}(2k^0)^{1/2}}\ [\ \bar{c}^\dagger(k)e^{ik\cdot x} + \bar{c}(k)e^{-ik\cdot x}\ ],
\end{array}
\tag{4.1}
$$

which corresponds to the equations of motion $\Box A_\mu = \Box C = \Box\bar{C} = 0$ obeyed by the basic fields of the theory. Here k_μ are the 2D momenta with their components $k_\mu = (k_0, k = k_1)$ and $k^2 = k_0^2 - k^2 = 0$ for the consistency with the above equations of motion. All the dagger operators are the creation operators and the non-dagger operators correspond to the annihilation operators for the basic quanta of the fields. The on-shell nilpotent version of the (co-)BRST symmetries (3.3) and (3.2) can be expressed, due to (3.16), as [9,37]

$$\begin{array}{ll} [Q_d, a^\dagger_\mu(k)] = -\varepsilon_{\mu\nu} k^\nu \bar{c}^\dagger(k), & [Q_d, a_\mu(k)] = \varepsilon_{\mu\nu} k^\nu \bar{c}(k), \\ \{Q_d, c^\dagger(k)\} = -i\varepsilon^{\mu\nu} k_\mu a^\dagger_\nu, & \{Q_d, c(k)\} = +i\varepsilon^{\mu\nu} k_\mu a_\nu, \\ \{Q_d, \bar{c}^\dagger(k)\} = 0, & \{Q_d, \bar{c}(k)\} = 0, \end{array} \tag{4.2}$$

$$\begin{array}{ll} [Q_b, a^\dagger_\mu(k)] = k_\mu c^\dagger(k), & [Q_b, a_\mu(k)] = -k_\mu c(k), \\ \{Q_b, c^\dagger(k)\} = 0, & \{Q_b, c(k)\} = 0, \\ \{Q_b, \bar{c}^\dagger(k)\} = ik^\mu a^\dagger_\mu(k), & \{Q_b, \bar{c}(k)\} = -ik^\mu a_\mu(k). \end{array} \tag{4.3}$$

Similar kinds of (anti-)commutation relations can be obtained with the anti-BRST and anti-co-BRST generators but we do not require them for our present analyses and discussions. For aesthetic reasons, we can define the most symmetric physical vacuum ($|vac>$) of the present theory as

$$\begin{array}{lll} Q_{(a)b} \; |vac>= 0, & Q_{(a)d} \; |vac>= 0, & Q_w \; |vac>= 0, \\ a_\mu(k) \; |vac>= 0, & c(k) \; |vac>= 0, & \bar{c}(k) \; |vac>= 0. \end{array} \tag{4.4}$$

In the above, it is clear that the physical vacuum is (anti-)BRST and (anti-)co-BRST invariant which imply the invariance w.r.t. Q_w as well. It is lucid now that a single photon state with polarization $e_\mu(k)$ and momenta k_μ can be created from the physical vacuum by the application of a creation operator $a^\dagger_\mu(k)$ as [37]

$$e^\mu a^\dagger_\mu(k) \; |vac> \equiv |e, vac>, \qquad k^\mu a^\dagger_\mu(k) \; |vac> \equiv |k, vac>= -i\{Q_b \bar{c}^\dagger(k)\} \; |vac>, \tag{4.5}$$

where the latter state $|k, vac>$ with momenta k_μ has been expressed by exploiting the anti-commutator $\{Q_b, \bar{c}^\dagger(k)\} = ik^\mu a^\dagger_\mu(k)$ from (4.3). Exploiting the $U(1)$ gauge transformation on the polarization vector $e_\mu(k) \to e'_\mu(k) = e_\mu(k) + iAk_\mu$, where A is a complex number, it is straightforward to check that

$$|e + i \; A \; k, vac>= |e, vac> + \; Q_b \; (A \; \bar{c}^\dagger(k))|vac>, \qquad Q_b \; |vac>= 0. \tag{4.6}$$

Thus, we conclude that a gauge transformed state for an original single photon state (i.e. $e^\mu(k) a^\dagger_\mu(k)|vac>$ with the polarization vector $e_\mu(k)$) is equal to the sum of the original state $|e, vac>$ plus a BRST exact state. In more sophisticated language, the gauge transformed state and the original state belong to the same cohomology class w.r.t. the conserved and nilpotent BRST charge Q_b. Similarly, the dual gauge transformation on the polarizations vector (i.e. $e_\mu(k) \to e'_\mu(k) = e_\mu(k) + iB\varepsilon_{\mu\nu}k^\nu \equiv e_\mu(k) + iB\tilde{k}_\mu$ where $\tilde{k}_\mu = \varepsilon_{\mu\nu}k^\nu$ and B is a complex number) will correspond to the following expression

$$|e + i \; B\tilde{k}, vac>= |e, vac> + \; Q_d \; (B \; c^\dagger(k)) \; |vac>, \qquad Q_d \; |vac>= 0, \tag{4.7}$$

where we have used the anti-commutator $\{Q_d, c^\dagger(k)\} = -i\varepsilon^{\mu\nu} k_\mu a^\dagger_\nu$. The above equation also implies that the dual-gauge transformed state is equal to the sum of the original state and a BRST co-exact state. With the four nilpotent and conserved charges $Q_{(a)b}, Q_{(a)d}$ and a bosonic conserved charge Q_w in the theory, the most symmetric physical state ($|phy>$) can be defined as

$$Q_{(a)b}\ |phy>= 0, \qquad Q_{(a)d}\ |phy>= 0, \qquad Q_w\ |phy>= 0. \tag{4.8}$$

Applying this physicality condition on the single photon state, we obtain the following relationships by exploiting the commutators $[Q_b, a^\dagger_\mu(k)] = k_\mu c^\dagger(k), [Q_d, a^\dagger_\mu(k)] = -\varepsilon_{\mu\nu} k^\nu \bar{c}^\dagger(k)$

$$\begin{array}{lll} Q_b\ |e + i\ A\ k, vac> & = & Q_b\ |e, vac> \equiv (k \cdot e)\ c^\dagger(k)\ |vac>= 0, \qquad (Q_b^2 = 0), \\ Q_d|e + i\ B\ \tilde{k}, vac> & = & Q_d\ |e, vac> \equiv (-\varepsilon^{\mu\nu} e_\mu k_\nu)\ \bar{c}^\dagger(k)\ |vac>= 0, \quad (Q_d^2 = 0), \end{array} \tag{4.9}$$

which imply the *transversality* (i.e. $k \cdot e = 0$) and an extra condition ($\varepsilon^{\mu\nu} e_\mu k_\nu = 0$) (that turns out to be useful in the proof for the topological nature of the 2D free Abelian gauge theory [9,10]) on the 2D photon because of the fact that $c^\dagger(k)|vac> \neq 0, \bar{c}^\dagger(k)|vac> \neq 0$. It is also obvious from the above discussion that for a single photon, *not* satisfying the above transversality and an extra condition illustrated in (4.9), the single (anti-)ghost state(s) ($\bar{c}^\dagger(k)|vac>, c^\dagger(k)|vac>$) created by the operators $\bar{c}^\dagger(k)$ and $c^\dagger(k)$ would turn out to be BRST (co-)exact states. This explains the *no-(anti-)ghost* theorem in the context of the BRST cohomology. Physically, it amounts to the well-known fact (see, e.g., [42]) that the contributions coming from the longitudinal and scalar degrees of freedom of the photons, at any arbitrary order of the perturbation theory calculations, are cancelled by the presence of (anti-)ghost fields. Ultimately, the physicality criteria $Q_b|e, vac>= 0, Q_d|e, vac>= 0, Q_w|e, vac>= 0$ on a single photon state implies the transversality and masslessness of the photon because of the following *mutually consistent* relationships that emerge from the above condition with conserved and local charges (see, e.g., [9] for more detail)

$$\begin{array}{l} Q_b\ |e, vac>= 0\ \rightarrow\ (k \cdot e) = 0, \\ Q_d\ |e, vac>= 0\ \rightarrow\ (\varepsilon_{\mu\nu} e^\mu k^\nu) = 0, \\ Q_w\ |e, vac>= 0,\ \rightarrow\ k^2 = 0. \end{array} \tag{4.10}$$

A thorough and complete discussion on this result and its implication to the topological nature of the theory (where there are no propagating degrees of freedom for the $U(1)$ gauge field A_μ) can be found in our earlier works (see, e.g., [9,10]). In the language of the Hodge decomposition theorem, it can be seen that the masslessness condition ($k^2 = 0$) coming from $Q_w|e, vac>= 0$ (which is the analogue of the action of the Laplacian operator on the harmonic state) has the solutions $k \cdot e = 0$ and $\varepsilon_{\mu\nu} e^\mu k^\nu = 0$ that are given by the top two equations (i.e., the analogues of the operation of the (co-)exterior derivatives).

With our understanding of the 2D free Abelian gauge theory as the background, we shall dwell a bit on the BRST cohomology connected with the 4D free Abelian 2-form gauge theory. For this purpose, first, we begin with the definition of the physical state $|phys>$ and the physical vacuum ($|vacm>$) of the 4D theory as

$$Q_{(A)B}\ |phys>= 0, \qquad Q_{(A)D}\ |phys>= 0, \qquad Q_W\ |phys>= 0, \tag{4.11}$$

$$
\begin{array}{llll}
Q_{(A)B}|vacm>=0, & Q_{(A)D}|vacm>=0, & Q_W|vacm>=0, & \\
b_{\mu\nu}(k)\;|vacm>=0, & c_\mu(k)\;|vacm>=0, & \bar{c}_\mu(k)\;|vacm>=0, & \\
\bar{b}(k)\;|vacm>=0, & f_1(k)\;|vacm>=0, & f_2(k)\;|vacm>=0, & b(k)|vacm>=0,
\end{array}
\qquad (4.12))
$$

where *the on-shell nilpotent* (anti-)BRST charges $Q_{(A)B}$ and (anti-)co-BRST charges $Q_{(A)D}$ are the generators of the corresponding transformations (3.13) and (3.14) for the Lagrangian density (3.12). Here the bosonic charge $Q_W = \{Q_B, Q_D\} = \{Q_{AB}, Q_{AD}\}$ and the rest of the annihilation operators in the above are from the normal mode expansion of the basic fields of the theory that are present in the Lagrangian density (3.12). These expansions are as follows

$$
\begin{array}{lcl}
B_{\mu\nu}(x) &=& \int \frac{d^3k}{(2\pi)^{3/2}(2k^0)^{3/2}}\;[\;b^\dagger_{\mu\nu}(k)e^{ik\cdot x} + b_{\mu\nu}(k)e^{-ik\cdot x}\;], \\
C_\mu(x) &=& \int \frac{d^3k}{(2\pi)^{3/2}(2k^0)^{3/2}}\;[\;c^\dagger_\mu(k)e^{ik\cdot x} + c_\mu(k)e^{-ik\cdot x}\;], \\
\bar{C}_\mu(x) &=& \int \frac{d^3k}{(2\pi)^{3/2}(2k^0)^{3/2}}\;[\;\bar{c}^\dagger_\mu(k)e^{ik\cdot x} + \bar{c}_\mu(k)e^{-ik\cdot x}\;], \\
\bar{\beta}(x) &=& \int \frac{d^3k}{(2\pi)^{3/2}(2k^0)^{3/2}}\;[\;\bar{b}^\dagger(k)e^{ik\cdot x} + \bar{b}(k)e^{-ik\cdot x}\;], \\
\beta(x) &=& \int \frac{d^3k}{(2\pi)^{3/2}(2k^0)^{3/2}}\;[\;b^\dagger(k)e^{ik\cdot x} + b(k)e^{-ik\cdot x}\;], \\
\phi_1(x) &=& \int \frac{d^3k}{(2\pi)^{3/2}(2k^0)^{3/2}}\;[\;f^\dagger_1(k)e^{ik\cdot x} + f_1(k)e^{-ik\cdot x}\;], \\
\phi_2(x) &=& \int \frac{d^3k}{(2\pi)^{3/2}(2k^0)^{3/2}}\;[\;f^\dagger_2(k)e^{ik\cdot x} + f_2(k)e^{-ik\cdot x}\;],
\end{array}
\qquad (4.13)
$$

where the four momenta k_μ have the components $k_\mu = (k_0, k_i)$ with $i = 1, 2, 3$ and the volume element $d^3k = dk_1 dk_2 dk_3$. All the individual dagger operators are the creation operator for a single quantum (particle) of the corresponding field (and the non-dagger operators are the annihilation operators). Exploiting the general definition of (3.16), we obtain the (anti-)commutators with the BRST charge Q_B as

$$
\begin{array}{ll}
[Q_B, b^\dagger_{\mu\nu}(k)] = (k_\mu c^\dagger_\nu(k) - k_\nu c^\dagger_\mu(k)), & [Q_B, b_{\mu\nu}(k)] = -(k_\mu c_\nu(k) - k_\nu c_\mu(k)), \\
[Q_B, b^\dagger(k)] = 0, & [Q_B, b(k)] = 0, \\
\{Q_B, \bar{c}^\dagger_\mu(k)\} = -k^\nu b^\dagger_{\mu\nu}(k) + k_\mu f^\dagger_1(k), & \{Q_B, \bar{c}_\mu(k)\} = k^\nu b_{\mu\nu}(k) - k_\mu f_1(k), \\
\{Q_B, c^\dagger_\mu(k)\} = -k_\mu b^\dagger(k), & \{Q_B, c_\mu(k)\} = +k_\mu b(k), \\
[Q_B, f^\dagger_2(k)] = 0, & [Q_B, f_2(k)] = 0, \\
[Q_B, f^\dagger_1(k)] = \frac{1}{2}k^\mu c^\dagger_\mu(k), & [Q_B, f_1(k)] = -\frac{1}{2}k^\mu c_\mu(k), \\
[Q_B, \bar{b}^\dagger(k)] = \frac{1}{2}k^\mu \bar{c}^\dagger_\mu(k), & [Q_B, \bar{b}(k)] = -\frac{1}{2}k^\mu \bar{c}_\mu(k),
\end{array}
\qquad (4.14)
$$

and the corresponding (anti-)commutators with the co-BRST charge Q_D are

$$
\begin{array}{ll}
[Q_D, b^\dagger_{\mu\nu}(k)] = \varepsilon_{\mu\nu\eta\zeta} k^\eta (\bar c^\zeta)^\dagger(k), & [Q_D, b_{\mu\nu}(k)] = -\varepsilon_{\mu\nu\eta\zeta} k^\eta \bar c^\zeta(k), \\
[Q_D, \bar b^\dagger(k)] = 0, & [Q_D, \bar b(k)] = 0, \\
\{Q_D, \bar c^\dagger_\mu(k)\} = k_\mu \bar b^\dagger(k), & \{Q_D, \bar c_\mu(k)\} = -k_\mu \bar b(k), \\
\{Q_D, c^\dagger_\mu(k)\} = k_\mu f^\dagger_2(k) - \frac{1}{2}\varepsilon_{\mu\nu\kappa\sigma} k^\nu (b^{\kappa\sigma})^\dagger(k), & \\
\{Q_B, c_\mu(k)\} = -k_\mu f_2(k) + \frac{1}{2}\varepsilon_{\mu\nu\kappa\sigma} k^\nu b^{\kappa\sigma}(k), & \\
[Q_D, f^\dagger_1(k)] = 0, & [Q_D, f_1(k)] = 0, \\
[Q_D, f^\dagger_2(k)] = \frac{1}{2} k^\mu \bar c^\dagger_\mu(k), & [Q_D, f_2(k)] = -\frac{1}{2} k^\mu \bar c_\mu(k), \\
[Q_D, b^\dagger(k)] = -\frac{1}{2} k^\mu c^\dagger_\mu(k), & [Q_D, \bar b(k)] = +\frac{1}{2} k^\mu c_\mu(k).
\end{array}
\tag{4.15}
$$

A few comments are in order now. First of all, it can be seen that the expansions for the vector anti-commuting (anti-)ghost fields $(\bar C_\mu)C_\mu$ in (4.13) are not consistent with the equations of motion $\Box C_\mu = \frac{3}{2}\,\partial_\mu(\partial\cdot C)$, $\Box \bar C_\mu = \frac{3}{2}\,\partial_\mu(\partial\cdot \bar C)$ unless we choose a gauge such that $(\partial\cdot C) = (\partial\cdot\bar C) = 0$ thereby implying $\Box C_\mu = \Box\bar C_\mu = 0$. Second, the above choice will imply $k^2 = 0,\ k\cdot c^\dagger = k\cdot c = 0,\ k\cdot\bar c^\dagger = k\cdot\bar c = 0$ in the (momentum) phase space. Third, it is evident that the last four commutators of (4.14) and (4.15) will turn out to be zero for the above gauge choice. We shall come to these points in a more detailed fashion in the next section where we will discuss the connection between little group and cohomology.

Let us now concentrate on a single particle quantum state for the 2-form field with polarization tensor $e_{\mu\nu}(k) = -e_{\nu\mu}(k)$. This can be created from the vacuum state by application of the creation operator $b^\dagger_{\mu\nu}(k)$ as

$$
e^{\mu\nu}(k) b^\dagger_{\mu\nu}(k) |vacm> \equiv |\tilde e, vacm> . \tag{4.16}
$$

The physicality criteria on this single particle quantum state for the 2-form field leads to

$$
\begin{array}{lcl}
Q_B\, |\tilde e, vacm> = 0 & \rightarrow & [Q_B, e^{\mu\nu}(k) b^\dagger_{\mu\nu}(k)]\ |vacm> = 0, \\
Q_D\, |\tilde e, vacm> = 0 & \rightarrow & [Q_D, e^{\mu\nu}(k) b^\dagger_{\mu\nu}(k)]\ |vacm> = 0, \\
Q_W\, |\tilde e, vacm> = 0 & \rightarrow & [Q_W, e^{\mu\nu}(k) b^\dagger_{\mu\nu}(k)]\ |vacm> = 0.
\end{array}
\tag{4.17}
$$

Exploiting the following commutation relations

$$
\begin{array}{l}
[Q_B, b^\dagger_{\mu\nu}(k)] = (k_\mu c^\dagger_\nu(k) - k_\nu c^\dagger_\mu(k)), \qquad [Q_D, b^\dagger_{\mu\nu}(k)] = \varepsilon_{\mu\nu\eta\xi} k^\eta (\bar c^\xi)^\dagger(k), \\
[Q_W, b^\dagger_{\mu\nu}(k)] = \frac{i}{2}\ [\ \varepsilon_{\mu\nu\zeta\eta} k_\sigma - \varepsilon_{\mu\nu\zeta\sigma} k_\eta + \varepsilon_{\nu\zeta\sigma\eta} k_\mu - \varepsilon_{\mu\zeta\sigma\eta} k_\nu\ ]\ k^\zeta (b^{\sigma\eta})^\dagger(k),
\end{array}
\tag{4.18}
$$

where the top two equations have been taken from (4.14) and (4.15) and the last commutation relation has been extracted from the following bosonic symmetry transformations $s_W = \{s_B, s_D\}$ for the Lagrangian density (3.12)

$$
\begin{array}{l}
s_W B_{\mu\nu} = \varepsilon_{\mu\nu\eta\zeta}\partial^\eta(\partial_\sigma B^{\sigma\zeta}) + \frac{1}{2}\varepsilon_{\nu\zeta\sigma\eta}\partial_\mu(\partial^\zeta B^{\sigma\eta}) - \frac{1}{2}\varepsilon_{\mu\zeta\sigma\eta}\partial_\nu(\partial^\zeta B^{\sigma\eta}), \\
s_W C_\mu(x) = -\frac{1}{2}\partial_\mu(\partial\cdot C), \quad s_W \bar C_\mu(x) = +\frac{1}{2}\partial_\mu(\partial\cdot\bar C), \quad s_W(\phi_1, \phi_2, \beta, \bar\beta) = 0.
\end{array}
\tag{4.19}
$$

These can be recast in terms of the commutation relations between Q_W and the creation and annihilation operators for the fields as

$$
\begin{array}{ll}
[Q_W, b^\dagger_{\mu\nu}(k)] = \frac{i}{2}\ [\ \varepsilon_{\mu\nu\zeta\eta} k_\sigma - \varepsilon_{\mu\nu\zeta\sigma} k_\eta + \varepsilon_{\nu\zeta\sigma\eta} k_\mu - \varepsilon_{\mu\zeta\sigma\eta} k_\nu\ ]\ k^\zeta\ (b^{\sigma\eta})^\dagger(k), & \\
[Q_W, b_{\mu\nu}(k)] = \frac{i}{2}\ [\ \varepsilon_{\mu\nu\zeta\eta} k_\sigma - \varepsilon_{\mu\nu\zeta\sigma} k_\eta + \varepsilon_{\nu\zeta\sigma\eta} k_\mu - \varepsilon_{\mu\zeta\sigma\eta} k_\nu\ ]\ k^\zeta\ b^{\sigma\eta}(k), & \\
[Q_W, c^\dagger_\mu(k)] = -\frac{i}{2} k_\mu\ (k^\nu c^\dagger_\nu(k)), & [Q_W, c_\mu(k)] = +\frac{i}{2} k_\mu\ (k^\nu c_\nu(k)), \\
[Q_W, \bar c^\dagger_\mu(k)] = +\frac{i}{2} k_\mu\ (k^\nu \bar c^\dagger_\nu(k)), & [Q_W, \bar c_\mu(k)] = -\frac{i}{2} k_\mu\ (k^\nu \bar c_\nu(k)).
\end{array}
\tag{4.20}
$$

It is evident that the physicality conditions in (4.17) (which are the analogues of (4.10) for the 2D free Abelian theory) lead to the following restrictions

$$
\begin{array}{l}
e^{\mu\nu}(k)\;[\;k_\mu c^\dagger_\nu(k) - k_\nu c^\dagger_\mu(k)\;]\;|vacm>= 0 \rightarrow -2\;[e^{\mu\nu}(k) k_\nu)]\;c^\dagger_\mu(k)\;|vacm>= 0,\\
e^{\mu\nu}(k)\;[\;\varepsilon_{\mu\nu\eta\zeta} k^\eta\;(\bar{c}^\zeta)^\dagger(k)\;]\;|vacm>= 0 \rightarrow [\;\varepsilon_{\mu\nu\eta\zeta} e^{\mu\nu}(k) k^\eta\;]\;(\bar{c}^\zeta)^\dagger(k)\;|vacm>= 0,\\
\frac{i}{2}\;e^{\mu\nu}(k)\;[\;\varepsilon_{\mu\nu\zeta\eta} k_\sigma - \varepsilon_{\mu\nu\zeta\sigma} k_\eta + \varepsilon_{\nu\zeta\sigma\eta} k_\mu - \varepsilon_{\mu\zeta\sigma\eta} k_\nu\;]\;k^\zeta\;(b^{\sigma\eta})^\dagger(k)\;|vacm>= 0\\
\rightarrow i[\;\varepsilon_{\mu\nu\zeta\eta} k_\sigma - \varepsilon_{\mu\zeta\sigma\eta} k_\nu\;]\;e^{\mu\nu}(k)\;k^\zeta\;(b^{\sigma\eta})^\dagger(k)\;|vacm>= 0.
\end{array}
\tag{4.21}
$$

The top equation establishes the transversality condition for the 2-form gauge field because $k_\mu \varepsilon^{\mu\nu} \equiv -\varepsilon^{\mu\nu} k_\nu = 0$ shows that the polarization tensor and momenta are orthogonal to one-another. We draw this conclusion because $c^\dagger_\mu(k)\;|vacm>\neq 0$ and it actually creates a quantum of the ghost field $C_\mu(x)$. For the transversality *not* to be satisfied, the above relation demonstrates the famous *no-ghost theorem* in the language of the BRST cohomology in the sense that the state $c^\dagger_\mu(k)\;|vacm>$ is a BRST exact state and, hence, a cohomologically trivial state. The next condition due to the physicality criteria (w.r.t. the conserved and nilpotent co-BRST charge $Q_D|phys>= 0$) finally implies the dual-transversality condition: $\varepsilon_{\mu\nu\kappa\zeta} e^{\nu\kappa}(k) k^\zeta = 0$. This relation is, in some sense, an extension of the the second equation of (4.10) (valid for the 2D free Abelian gauge theory) to the 4D free Abelian 2-form gauge theory. This also demonstrates the fact that if $\varepsilon_{\mu\nu\kappa\zeta} e^{\nu\kappa}(k) k^\zeta \neq 0$, the one-particle vector anti-ghost state $\bar{c}^\dagger_\mu(k)\;|vacm>$ is a BRST co-exact state and, therefore, a trivial state as far as the BRST cohomology, HDT and choice of the physical state to be the harmonic state are concerned. This establishes the no-anti-ghost theorem (without resorting to the implementation of the anti-BRST charge (Q_{AB}) in the physicality criteria which also leads to the same conclusion). The last condition of (4.21), with the help of the transversality condition $-k_\mu e^{\mu\nu}(k) = e^{\mu\nu}(k) k_\nu = 0$, leads to the masslessness (i.e. $\Box B_{\mu\nu}(x) = 0 \rightarrow k^2 = k_0^2 - k_i^2 = 0$) condition for the 2-form $B_{\mu\nu}$ field when we expand the whole equation in terms of the physical components of k^μ and $e^{\mu\nu}(k)$. Thus, ultimately, the criteria in (4.21) physically imply the *transversality* and *masslessness* of the free Abelian 2-form gauge field.

5 Wigner's little group: (dual-)gauge transformations and cohomology

We begin with the most general form of the Wigner's little group matrix $\{W^\mu_\nu(\theta, u, v)\}$ for a massless particle moving along the z-direction of the 4D spacetime manifold as [30-35]

$$
\{W(\theta,u,v)\} = \begin{pmatrix}
(1+\dfrac{u^2+v^2}{2}) & (u cos\theta - v sin\theta) & (u sin\theta + v cos\theta) & -(\dfrac{u^2+v^2}{2})\\
u & cos\theta & sin\theta & -u\\
v & -sin\theta & cos\theta & -v\\
(\dfrac{u^2+v^2}{2}) & (u cos\theta - v sin\theta) & (u sin\theta + v cos\theta) & (1-\dfrac{u^2+v^2}{2})
\end{pmatrix},
\tag{5.1}
$$

where θ is the rotational parameter and u, v are the translational parameters defining $T(2)$ in the xy plane. By definition, this matrix preserves the four momentum $k^\mu = (\omega, 0, 0, \omega)^T$

of a massless ($k^2 = 0$) particle with energy ω and it can be factorized elegantly as

$$(k^\mu) \to (k^\mu)' = W^\mu_\nu\ (k^\nu) = (k^\mu), \qquad W(\theta, u.v) = R(\theta)\ W(0, u, v). \tag{5.2}$$

The matrix $R(\theta)$ in the above represents the rotation about the z-axis

$$R(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos\theta & sin\theta & 0 \\ 0 & -sin\theta & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.3}$$

and the matrix $\{W(0, u, v)\}$ is found to be isomorphic to the two parameter translation group $T(2)$ (i.e. $T(2) \sim W(0, u, v)$) in the two-dimensional Euclidean plane (xy) which is a plane perpendicular to the propagation of the light-like (massless) particle along the z-direction. *It is crystal clear that there exists no such kind of plane for the discussion of any arbitrary gauge theory in* $(1 + 1)$*-dimension (2D) of spacetime.*

Now let us concentrate on the gauge transformation (2.6) on the 2-form free Abelian basic gauge field $B_{\mu\nu}$ (i.e. $B_{\mu\nu} \to B^{(g)}_{\mu\nu} = B_{\mu\nu} + (\partial_\mu \alpha_\nu - \partial_\nu \alpha_\mu)$). This transformation, in the language of the anti-symmetric ($e_{\mu\nu}(k) = -e_{\nu\mu}(k)$) polarization tensor can be expressed, in its contravariant form, as [36]

$$e^{\mu\nu}(k) \to (e^{\mu\nu})^{(g)}(k) = e^{\mu\nu}(k) + i\ [k^\mu \alpha^\nu(k) - k^\nu \alpha^\mu(k)]. \tag{5.4}$$

For the massless ($k^2 = 0$) 2-form $B_{\mu\nu}(x)$ field, it can be seen, from the physicality condition (4.21) (with the BRST charge Q_B) that the momenta and the polarization tensor are orthogonal to one-another: $k_\mu e^{\mu\nu}(k) = -e^{\mu\nu}(k) k_\nu = 0$. This orthogonality condition can also be obtained from the equation of motion for the massless (i.e. $k^2 = 0$) 2-form gauge field $B_{\mu\nu}(x)$ when the Lagrangian density contains only the kinetic energy term. One has to put masslessness ($k^2 = 0$) condition, however, by hand in the latter case [36]. This transversality condition reduces the six independent components of the anti-symmetric 4D tensor $e^{\mu\nu}(k)$ to three only (cf. (5.7) below). For this to be seen explicitly, we have the following matrix product

$$k_\mu e^{\mu\nu} = 0 \Rightarrow (\omega, 0, 0, -\omega) \begin{pmatrix} 0 & e^{01} & e^{02} & e^{03} \\ -e^{01} & 0 & e^{12} & e^{13} \\ -e^{02} & -e^{12} & 0 & e^{23} \\ -e^{03} & -e^{13} & -e^{23} & 0 \end{pmatrix} = 0, \tag{5.5}$$

where the covariant version of momentum 4-vector is taken to be $k_\mu = (\omega, 0, 0, -\omega)^T$ for our choice of the contravariant momentum vector $k^\mu = (\omega, 0, 0, \omega)^T$. The above conditions yield the following relationships among the components of the polarization matrix $\{e^{\mu\nu}(k)\}$

$$e^{03} = 0, \qquad e^{01} + e^{13} = 0, \qquad e^{02} + e^{23} = 0. \tag{5.6}$$

Thus, the above transversality condition (5.5) reduces the polarization tensor to

$$\{e^{\mu\nu}(k)\}_{(r)} = \begin{pmatrix} 0 & e^{01} & e^{02} & 0 \\ -e^{01} & 0 & e^{12} & -e^{01} \\ -e^{02} & -e^{12} & 0 & -e^{02} \\ 0 & e^{01} & e^{02} & 0 \end{pmatrix}. \tag{5.7}$$

Yet another reduction of the anti-symmetric tensor $\{e^{\mu\nu}(k)\}$ can be achieved by the choice of the infinitesimal gauge parameters α's of the transformation (5.4) for our choice of the 4-momenta $k^\mu = (\omega, 0, 0, \omega)^T$. For instance, it can be checked that, for the following choice of the local gauge parameters

$$\alpha^1(k) = \frac{i}{\omega}\, e^{01}(k), \qquad \alpha^2(k) = \frac{i}{\omega}\, e^{02}(k), \tag{5.8}$$

one can gauge away the components $e^{01}(k)$ and $e^{02}(k)$ (i.e. $(e^{01})(k) \to (e^{01}(k))^{(g)} = 0$, $(e^{02})(k) \to (e^{02}(k))^{(g)} = 0$). Even through $e^{03}(k) = 0$ due to (5.6), in the general gauge transformation (5.4) one has to choose $\alpha^0(k) = \alpha^3(k)$ for the identity (i.e. $0 = 0$) corresponding to the gauge transformation on $e^{03}(k)$, to be satisfied. Thus, ultimately, we are left with the anti-symmetric second-rank matrix $\{e^{\mu\nu}(k)\}$ with only one degree of freedom as

$$\{e^{\mu\nu}(k)\}_{(R)} = \; e^{12}(k) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.9}$$

At this juncture, we can exploit the crucial role played by the translation subgroup $T(2) \sim W(0, u, v)$ of the Wigner's little group in generating the gauge transformation for the polarization tensor as follows

$$\begin{aligned} e^{\mu\nu}(k) &\to (e^{\mu\nu})'(k) = (W)^\mu_\sigma \; (e^{\sigma\kappa}(k)) \; (W)^\nu_\kappa \equiv W(0, u, v) \cdot e(k) \cdot W(0, u, v)^T, \\ e^{\mu\nu}(k) &\to (e^{\mu\nu})'(k) = e^{12}(k) \begin{pmatrix} 0 & -v & u & 0 \\ v & 0 & 1 & v \\ -u & -1 & 0 & -u \\ 0 & -v & u & 0 \end{pmatrix}, \end{aligned} \tag{5.10}$$

where the matrix product has been denoted by the dot $(\cdot)$ product. The above transformation can be re-expressed as

$$\{e^{\mu\nu}(k)\} \to \{(e^{\mu\nu})'(k)\} = \{e^{\mu\nu}(k)\}_{(R)} + e^{12}(k) \begin{pmatrix} 0 & -v & u & 0 \\ v & 0 & 0 & v \\ -u & 0 & 0 & -u \\ 0 & -v & u & 0 \end{pmatrix}, \tag{5.11}$$

where the untransformed matrix $\{e^{\mu\nu}(k)\}_{(R)}$ is given in (5.9). This is obviously a gauge transformation corresponding to (5.4) for the following relationships among the parameters of the gauge symmetry group and the translation $T(2)$ subgroup of the Wigner's little group

$$\alpha^1(k) = \frac{iv}{\omega}\, e^{12}(k), \qquad \alpha^2(k) = -\frac{iu}{\omega}\, e^{12}(k), \qquad \alpha^3 = \alpha^0. \tag{5.12}$$

It will be noted that the last condition comes from the consideration of the gauge transformation (5.4) on the component $e^{03}(k)$ (or $e^{30}(k)$). The other two conditions emerge from the gauge transformations on $e^{01}(k), e^{02}(k), e^{13}(k), e^{23}(k)$.... etc. This establishes the connection between translation subgroup $T(2) \sim W(0,u,v)$ of the Wigner's little group and the gauge transformation defined in (2.6).

Now we dwell a bit on the origin of the dual-gauge transformation ($B_{\mu\nu}(x) \to B_{\mu\nu}^{(dg)}(x) = B_{\mu\nu}(x) + \varepsilon_{\mu\nu\kappa\xi}\partial^\kappa \Sigma^\xi(x)$) defined in (2.6) in the framework of Wigner's little group. It is clear that this transformation in the momentum space can be expressed, following [36], in terms of the transformation on the contravariant polarization anti-symmetric tensor $e^{\mu\nu}(k)$ as

$$(e^{\mu\nu})(k) \to (e^{\mu\nu})^{(dg)}(k) = (e^{\mu\nu})(k) + i\varepsilon^{\mu\nu\eta\xi} k_\eta \Sigma_\xi(k). \tag{5.13}$$

We would like to answer the question: in the framework of the translation subgroup $T(2)$ of the Wigner's little group, where is the existence of the dual-gauge transformation of (2.6) which has been expressed as (5.13)? It is evident that the argument, based on the *transversality* of the 2-form gauge field emerging from the physicality condition with the BRST charge Q_B, goes along the same lines till equation (5.7) where we have obtained the reduced polarization tensor $\{e^{\mu\nu}(k)\}_{(r)}$. It can be checked, using (5.13) and $k_\mu = (\omega, 0, 0, -\omega)^T$, that for the following choice of the infinitesimal dual-gauge parameters Σ's

$$\Sigma_1(k) = \frac{i}{\omega}\, e^{02}(k), \qquad \Sigma_2(k) = -\frac{i}{\omega}\, e^{01}(k), \tag{5.14}$$

one can gauge away the components $e^{01}(k)$ and $e^{02}(k)$ from the matrix $\{e^{\mu\nu}(k)\}_{(r)}$ (i.e. $(e^{01}(k) \to (e^{01})^{(dg)}(k) = 0,\ e^{02}(k) \to (e^{02})^{(dg)}(k) = 0)$. We note here that there is a kind of "duality" between (5.8) and (5.14) in the sense that

$$\alpha^1(k) \leftrightarrow -\Sigma_2(k), \qquad \alpha^2(k) \leftrightarrow \Sigma_1(k). \tag{5.15}$$

The relationship in (5.14) finally allows us to obtain the reduced matrix $\{e^{\mu\nu}(k)\}_{(R)}$ of (5.9). Having obtained (5.9) by exploiting the dual-gauge transformations (5.13), the rest of the calculations related to the transformation of the polarization tensor $e^{\mu\nu}(k)$ by the translation subgroup $T(2) \sim W(0,u,v)$ of the Wigner's little group, are exactly the same as (5.10) and (5.11). It is very interesting to note that (5.11), not only demonstrates the existence of the gauge transformations (5.4) with the choice of parameters α's in (5.12), but it also establishes the existence of the dual-gauge transformations of (5.13) with the following relationships among the dual-gauge parameters Σ's and the parameters of the translation $T(2)$ subgroup of the Wigner's little group

$$\Sigma_1(k) = -\frac{iu}{\omega}\, e^{12}(k), \qquad \Sigma_2(k) = -\frac{iv}{\omega}\, e^{12}(k), \qquad \Sigma_3(k) = -\Sigma_0(k), \tag{5.16}$$

where the last relation $\Sigma_3(k) = -\Sigma_0(k)$ emerges from the dual-gauge transformations (5.13) applied to the transformation for the polarization tensor component $e^{12}(k)$ (or $e^{21}(k)$). The expressions for the infinitesimal dual-gauge parameters $\Sigma_1(k)$ and $\Sigma_2(k)$ in the above equation are derived from the transformation properties of the polarization tensor components $e^{01}(k), e^{02}(k), e^{13}(k), e^{23}(k)$... etc. Moreover, the "duality" kind of transformations of (5.15) are reflected here too if we compare the relationships obtained in (5.12) and (5.16).

Now let us concentrate briefly on the other *independent* restriction (i.e. $\varepsilon_{\mu\nu\kappa\zeta}\, e^{\nu\kappa}(k)\, k^\zeta = 0$) that emerges from the physicality condition with the dual-BRST charge Q_D in (4.21). From this relation too, we can show that there is only a single degree of freedom associated with the 4D polarization tensor $e^{\mu\nu}(k)$ for the massless 2-form gauge field $B_{\mu\nu}(x)$. In contrast to the transversality condition (5.5) that has been used earlier to derive (5.6), it can be checked that for $\mu = 0, 1, 2, 3$, we obtain the following relationships among some of the six-independent components of the 4D polarization tensor matrix $\{e^{\mu\nu}(k)\}$

$$e^{12} = 0, \qquad e^{01} + e^{13} = 0, \qquad e^{02} + e^{23} = 0, \tag{5.17}$$

from the dual-transversality relationship $\varepsilon_{\mu\nu\kappa\zeta}\, e^{\nu\kappa}(k)\, k^\zeta = 0$ where $k^\mu = (\omega, 0, 0, \omega)^T$. Thus, the reduced form ($\{e^{\mu\nu}(k)\}_{(red)}$) of the polarization tensor is

$$\{e^{\mu\nu}(k)\}_{(red)} = \begin{pmatrix} 0 & e^{01} & e^{02} & e^{03} \\ -e^{01} & 0 & 0 & -e^{01} \\ -e^{02} & 0 & 0 & -e^{02} \\ -e^{03} & e^{01} & e^{02} & 0 \end{pmatrix}. \tag{5.18}$$

From the above reduced polarization tensor matrix $\{e^{\mu\nu}(k)\}_{(red)}$, one can gauge away $e^{01}(k)$ and $e^{02}(k)$ either (i) by the gauge transformations (5.4) with the choice of and relationship between the gauge parameters α's as

$$\alpha^1(k) = \frac{i}{\omega}\, e^{01}(k), \qquad \alpha^2(k) = \frac{i}{\omega}\, e^{02}(k), \qquad \alpha^0(k) = \alpha^3(k), \tag{5.19}$$

(where the last restriction emerges from the transformation of the component $e^{03}(k) = -e^{30}(k)$), or (ii) by choosing the dual-gauge parameters Σ's in transformations (5.13) as

$$\Sigma_1(k) = -\frac{i}{\omega}\, e^{02}(k), \qquad \Sigma_2(k) = +\frac{i}{\omega}\, e^{01}(k), \tag{5.20}$$

which is the analogue of (5.14) (with merely a sign difference). It is clear that $e^{12}(k) = 0$ due to (5.17). However, as far as the general dual-gauge transformation (5.13) is concerned, for the validity of the identity (i.e. $0 = 0$) corresponding to the transformations (5.13) on $e^{12}(k)$, one has to choose $\Sigma_3(k) = -\Sigma_0(k)$. Final reduced version of the polarization tensor (i.e. the analogue of (5.9)) with the restriction $\varepsilon_{\mu\nu\kappa\zeta}\, e^{\nu\kappa}(k)\, k^\zeta = 0$ is

$$\{e^{\mu\nu}(k)\}_{(Red)} = e^{03}(k) \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{5.21}$$

Now we can check the action of the translation subgroup $T(2) \sim W(0, u, v)$ on the reduced polarization tensor (5.21) in generating the (dual-)gauge transformations. It can be calculated explicitly to see that the analogue of the equation (5.11) is now

$$\{e^{\mu\nu}(k)\} \to \{(e^{\mu\nu})'(k)\} = \{e^{\mu\nu}(k)\}_{(Red)} + e^{03}(k) \begin{pmatrix} 0 & -v & -u & 0 \\ u & 0 & 0 & u \\ v & 0 & 0 & v \\ 0 & -v & -u & 0 \end{pmatrix}, \tag{5.22}$$

where $\{e^{\mu\nu}(k)\}_{(Red)}$ is defined in (5.21). This transformation corresponds to the gauge transformation (5.4) with the following choice of the infinitesimal gauge parameters α's

$$\alpha^1(k) = \frac{iu}{\omega}\, e^{03}(k), \qquad \alpha^2(k) = -\frac{iv}{\omega}\, e^{03}(k), \qquad \alpha^3(k) = \alpha^0(k), \tag{5.23}$$

where the last relation emerges from the transformation of the polarization component $e^{03}(k)$ (or $e^{30}(k)$) and the rest of the relationships emerge from the transformations of the components $e^{01}(k), e^{02}(k), e^{13}(k)$.. etc. It is extremely interesting to point out that the transformations (5.22) also contain the dual-gauge transformation of (5.13) for the following choice of the dual-gauge parameters

$$\Sigma_1(k) = -\frac{iv}{\omega}\, e^{03}(k), \qquad \Sigma_2(k) = -\frac{iu}{\omega}\, e^{03}(k). \tag{5.24}$$

Here the transformation property of the component $e^{12}(k)$ under the dual-gauge transformation (5.13) has not been discussed because $e^{12}(k) = 0$ due to (5.17). However, if one discusses it in the general framework of dual-gauge transformation (5.13), the corresponding identity (i.e. $0 = 0$) will be satisfied iff $\Sigma_3 = -\Sigma_0$. It is worth pointing out that (i) here there is no restriction like the last relationship of (5.16), and (ii) the duality of (5.15) do exist here too between (5.23) and (5.24).

At this stage, there are a few comments in order. First, it is unequivocally clear that the conditions (i) $e^{\mu\nu}(k)k_\nu = -k_\mu e^{\mu\nu}(k) = 0$, and (ii) $\varepsilon_{\mu\nu\kappa\zeta} e^{\nu\kappa}(k) k^\zeta = 0$ emerging from the physicality criteria (4.21) with the (co-)BRST charges ($Q_{(D)B}$) imply both the gauge and dual-gauge transformations which are primarily incorporated in the transformations generated by the translation $T(2)$ subgroup of the Wigner's little group. Thus, we note that the (dual-)gauge transformations, (co-)BRST transformations and the transformations generated by the translation subgroup $T(2)$ of the Wigner's little group are inter-related. Second, it can be seen from the equations (5.5), (5.6) and (5.9) that the transversality condition and the gauge (or BRST) transformations imply that only $e^{12}(k)$ is the single degree of freedom left for the 2-form field $B_{\mu\nu}(x)$. On top of it, if we apply the dual-transversality condition *independently*, we can get rid of $e^{12}(k)$ as well (cf. (5.17)). Thus, the 2-form field $B_{\mu\nu}(k)$ becomes topological in nature. In fact, it has been shown that the total (co-)BRST invariant theory, described by (3.12), is a quasi-topological theory [15] because, in addition to the topological field $B_{\mu\nu}$, the scalar fields ϕ_1, ϕ_2 and the ghost fields do exist in the theory. Third, we have not considered here the restrictions emerging from the bosonic conserved charge Q_W because it is automatically implied when both the conditions due to (co-)BRST charges ($Q_{(D)B}$) are taken into account together in an independent way.

Let us now come back to the discussion about our comments after equation (4.15). Exploiting the (anti-)commutators of (4.14) and (4.15) and invoking the nilpotency ($Q^2_{(D)B} = 0$) of the (co-)BRST charges ($Q_{(D)B}$), it can be checked that the following commutation relations

$$\begin{array}{ll} [Q_B, \{Q_B, \bar{c}^\dagger_\mu(k)\}] = 0, & [Q_B, \{Q_B, \bar{c}_\mu(k)\}] = 0, \\ [Q_D, \{Q_D, c^\dagger_\mu(k)\}] = 0, & [Q_D, \{Q_D, c_\mu(k)\}] = 0, \end{array} \tag{5.25}$$

are satisfied iff $k^2 = 0, k \cdot \bar{c}^\dagger = k \cdot \bar{c} = 0, k \cdot c^\dagger = k \cdot c = 0$. In fact, the masslessness condition $k^2 = 0$ is satisfied because of our choice of the momentum 4-vectors $k^\mu = (\omega, 0, 0, \omega)^T, k_\mu = (\omega, 0, 0, -\omega)^T$. However, the latter relations emerge from the condition $\partial_\mu C^\mu \equiv \partial^\mu C_\mu = (\partial \cdot C) = 0 \Rightarrow k \cdot c^\dagger = k \cdot c = 0$ and condition $\partial_\mu \bar{C}^\mu \equiv \partial^\mu \bar{C}_\mu = (\partial \cdot \bar{C}) = 0 \Rightarrow k \cdot \bar{c}^\dagger = k \cdot \bar{c} = 0$. We lay emphasis on the fact that *these conditions are met in our whole discussion.* In fact, it is clear that, in the BRST formalism, the polarization tensor $e^{\mu\nu}(k)$ transforms as

$$\begin{array}{l} e^{\mu\nu}(k) \rightarrow e^{\mu\nu}(k))^{(db)} = e^{\mu\nu}(k) + i \varepsilon^{\mu\nu\kappa\zeta} \partial_\kappa \bar{C}_\zeta(k), \\ e^{\mu\nu}(k) \rightarrow e^{\mu\nu}(k))^{(b)} = e^{\mu\nu}(k) + i \; [\; k^\mu C^\nu(k) - k^\nu C^\mu(k) \;], \end{array} \tag{5.26}$$

which are nothing but the generalizations of the (dual-)gauge transformations to (co-)BRST transformations (5.13) and (5.4) where $\alpha^\mu(k) \rightarrow C^\mu(k)$ and $\Sigma^\mu(k) \rightarrow \bar{C}^\mu(k)$. It is very clear and transparent now that the last relationships of the (dual-)gauge transformations (5.16) and (5.12) are replaced by the vector (anti-)ghost fields of the (co-)BRST transformations as

$$\bar{C}_3(k) = -\bar{C}_0(k), \qquad\qquad C^3(k) = C^0(k), \tag{5.27}$$

which very convincingly satisfy $k \cdot C(k) = k \cdot \bar{C}(k) = 0$ implying $k \cdot c^\dagger = k \cdot c = 0, \; k \cdot \bar{c}^\dagger = k \cdot \bar{c} = 0$ with our choice of the momentum vectors as $k^\mu = (\omega, 0, 0, \omega)^T, k_\mu = (\omega, 0, 0, -\omega)^T$. The above conclusions can be drawn from the (dual-)gauge transformations (5.24) and (5.23) as well. For the proof of $\bar{C}_3(k) = -\bar{C}_0(k)$, it will be worthwhile to go through the discussions after equation (5.24). Thus, it is obvious that all the crucial comments after (4.15) are justified and there is nothing in the theory that has been imposed from outside.

Now we would like to end this section with a few brief comments about the connection between the (dual-)gauge transformed states (that are also connected with the transformation on the polarization tensor $e^{\mu\nu}(k)$ by the translation $T(2)$ subgroup of the Wigner's little group) in the QHSS and the BRST cohomology w.r.t. the conserved and nilpotent (co-)BRST charges. It is evident that a single particle quantum state of the 2-form field (SPQS) with the polarization tensor $e^{\mu\nu}(k)$ can be created from the vacuum by the application of the creation operator $b^\dagger_{\mu\nu}(k)$ (i.e. $e^{\mu\nu}(k) b^\dagger_{\mu\nu}(k) \; |vacm>$) as given by (4.16). Exploiting the (anti-)commutation relations of (4.14), it can be checked that

$$\begin{array}{l} -2 \; [\; Q_B, (c^\mu)^\dagger(k) \bar{c}^\dagger_\mu(k) \;] = -2 \; \{Q_B, (c^\mu)^\dagger(k)\} \; \bar{c}^\dagger_\mu(k) + 2 (c^\mu)^\dagger(k) \; \{Q_B, \bar{c}^\dagger_\mu(k)\} \\ \equiv 2 b^\dagger(k) \; [k \cdot \bar{c}^\dagger(k)] + 2 \; [k \cdot c^\dagger(k)] \; f^\dagger_1(k) - 2 \; [k^\nu (c^\mu)^\dagger(k)] \; b^\dagger_{\mu\nu}(k). \end{array} \tag{5.28}$$

Applying our observation that $k \cdot \bar{c}^\dagger = k \cdot c^\dagger = 0$, it is straightforward to see that the above commutator, ultimately, reduces to the following expression (with $b^\dagger_{\mu\nu}(k) = -b^\dagger_{\nu\mu}(k)$)

$$-2 \; [\; Q_B, (c^\mu)^\dagger(k) \bar{c}^\dagger_\mu(k) \;] = +[\; k^\mu (c^\nu)^\dagger(k) - k^\mu (c^\mu)^\dagger(k) \;] \; b^\dagger_{\mu\nu}(k). \tag{5.29}$$

The physical implication of the above equation emerges when it is applied on the physical vacuum state (i.e. $Q_B \; |vacm >= 0$) of the theory as given below

$$\begin{aligned} &[\; k^\mu (c^\nu)^\dagger(k) - k^\mu (c^\mu)^\dagger(k) \;] \; b^\dagger_{\mu\nu}(k) \; |vacm >= -2 \; [\; Q_B, (c^\mu)^\dagger(k) \bar{c}^\dagger_\mu(k) \;] \; |vacm > \\ &\equiv Q_B(-2[\; (c^\mu)^\dagger(k) \bar{c}^\dagger_\mu(k) \;]) \; |vacm > . \end{aligned} \tag{5.30}$$

It is lucid and clear that the above state is a BRST exact state and, hence, in the language of HDT and BRST cohomology, it is a trivial state. Taking the sum of the equations (4.16) and (5.30) (with a factor of i), we obtain the following relationship between the gauge (i.e. BRST) transformed SPQS and the original SPQS (4.16)

$$\begin{aligned} &\Big(\; e^{\mu\nu}(k) + i \; [\; k^\mu (c^\nu)^\dagger(k) - k^\mu (c^\mu)^\dagger(k) \;] \; \Big) \; b^\dagger_{\mu\nu}(k) \; |vacm > \\ &= e^{\mu\nu}(k) \; b^\dagger_{\mu\nu}(k) \; |vacm > + Q_B(\; -2i[(c^\mu)^\dagger(k) \bar{c}^\dagger_\mu(k) \;] \;) \; |vacm > . \end{aligned} \tag{5.31}$$

This establishes the fact that a gauge (or BRST) transformed state in the quantum Hilbert space (that is connected with the transformation on the polarization tensor $e^{\mu\nu}(k)$ by the translation subgroup $T(2)$ of the Wigner's little group) is the sum of the original SPQS plus a BRST exact state. In more sophisticated language, the gauge (or BRST) transformed SPQS and the original SPQS belong to the same cohomology class w.r.t. the conserved ($\dot{Q}_B = 0$) and nilpotent ($Q_B^2 = 0$) BRST charge Q_B. Similarly, it can be checked, exploiting the (anti-)commutators of (4.15), that the dual-gauge (or co-BRST) transformed SPQS and the original SPQS in the quantum Hilbert space of states are related to each-other as

$$\begin{aligned} &\Big(\; e^{\mu\nu}(k) + i \; [\; \varepsilon^{\mu\nu\sigma\zeta} \; k_\sigma \bar{c}^\dagger_\zeta(k) \;] \; \Big) \; b^\dagger_{\mu\nu}(k) \; |vacm >= e^{\mu\nu}(k) \; b^\dagger_{\mu\nu}(k) \; |vacm > \\ &+ Q_D(\; -2i[(\bar{c}^\mu)^\dagger(k) c^\dagger_\mu(k) \;] \;) \; |vacm >, \end{aligned} \tag{5.32}$$

where we have used the following commutation relationship

$$\begin{aligned} &-2 \; [\; Q_D, (\bar{c}^\mu)^\dagger(k) c^\dagger_\mu(k) \;] = -2 \; \{Q_D, (\bar{c}^\mu)^\dagger(k)\} \; c^\dagger_\mu(k) + 2(\bar{c}^\mu)^\dagger(k) \; \{Q_D, c^\dagger_\mu(k)\} \\ &\equiv -2\bar{b}^\dagger(k) \; [k \cdot c^\dagger(k)] + 2 \; [k \cdot \bar{c}^\dagger(k)] \; f_2^\dagger(k) + \varepsilon_{\mu\nu\kappa\sigma} \; k^\mu \; (\bar{c}^\nu)^\dagger \; (b^{\kappa\sigma})^\dagger(k). \end{aligned} \tag{5.33}$$

This demonstrates that a dual-gauge (or co-BRST) transformed SPQS and the original SPQS belong to the same cohomology class w.r.t. the conserved ($\dot{Q}_D = 0$) and nilpotent ($Q_D^2 = 0$) co-BRST charge Q_D. Thus, it is obvious that if we take into account the following three basic ideas: (i) the BRST cohomology w.r.t. (co-)BRST charges (ii) the HDT applied to the states of the quantum Hilbert space, and (iii) the choice of the physical states (as well as the vacuum) to be the harmonic states of the HDT, it can be proven explicitly that the (dual-)gauge (or (co-)BRST) transformed SPQS (that corresponds to the transformations on $e^{\mu\nu}(k)$ by the translation subgroup $T(2)$ of the Wigner's little group) and the original SPQS belong to the same cohomology class w.r.t. (co-)BRST charges $Q_{(D)B}$.

As commented after the equation (5.3), it is unequivocally clear that the Wigner's little group for any arbitrary 2D gauge theory is trivial. In fact, the matrix representation for it becomes identity matrix (cf. (5.1)). Thus, the (dual-)gauge transformations, discussed in section 2, for the free 2D Abelian gauge theory can not be described in the framework of Wigner's little group. However, it can be discussed in the framework of BRST cohomology based on the constraint analysis of the theory. In fact, physically there is nothing in the free 2D Abelian gauge theory because of the fact that both the degrees of freedom of the photon can be gauged away by the (dual-)gauge (or (co-)BRST) transformations. In other words, both the components of the polarization vector $e_\mu(k)$ in 2D can be gauged away by the (dual-)gauge (or (co-)BRST) transformations. Thus, the arguments of the translation subgroup $T(2)$ of the Wigner's little group does not work for 2D free Abelian gauge theory.

6 Conclusions

In our present investigation, we have demonstrated an interesting connection between (i) the (dual-)gauge transformations on the polarization tensor $e^{\mu\nu}(k) = -e^{\nu\mu}(k)$ of the 4D free Abelian 2-form gauge field generated by the first-class constraints of the theory, and (ii) similar transformations generated by the (Abelian invariant) two-parameter translation subgroup $T(2)$ of the Wigner's little group. Both the above transformations are shown to be equivalent for certain specific relationships among the (dual-)gauge parameters of the internal symmetry transformation group and the parameters of the translation subgroup $T(2)$ of the Wigner's little group (which does not transform the momentum vector k_μ of the massless gauge particle). The latter parameters characterize the Euclidean plane (xy) which is perpendicular to the z-direction of the propagation of the massless ($k^2 = 0$) gauge particle. In the framework of the BRST cohomology and HDT applied to the QHSS, it is established in our present endeavour that the (dual-)gauge (or (co-)BRST) transformed states in the QHSS are the sum of the original (untransformed) states plus the (co-)BRST exact states. Thus, it becomes crystal clear that *the changes* in the original state due to the (dual-)gauge (or (co-)BRST) transformations correspond to the cohomologically *trivial* states as they satisfy trivially the physicality condition $Q_B|phys>= 0, Q_D|phys>= 0, Q_W|phys>= 0$ for our choice of the physical states to be the harmonic states. This happens primarily due to the nilpotency ($Q^2_{(D)B} = 0$) of the (co-)BRST charges ($Q_{(D)B}$) and the definition of the bosonic charge $Q_W = Q_B Q_D + Q_D Q_B$ (which turns out to be the analogue of the Laplacian operator). These statements can be proven in terms of the nilpotent and conserved anti-BRST charge (Q_{AB}) and anti-co-BRST charge (Q_{AD}) as well.

It is interesting to point out that, in the framework of BRST formalism, the equation of motion $\Box B_{\mu\nu} = 0$ for the 2-form gauge field is such that the masslessness condition ($k^2 = 0$) is implied automatically unlike the discussions in [36] where the massive and massless cases for the above field are considered separately. In fact, in [36], the Lagrangian density is not gauge-fixed to start with and it contains the kinetic energy term ($\mathcal{L}_0 = \frac{1}{12} H^{\mu\nu\kappa} H_{\mu\nu\kappa}$)

only. This is why the transversality condition $k_\mu \, e^{\mu\nu}(k) = -e^{\mu\nu}(k) \, k_\nu = 0$ emerges from the equation of motion when the masslessness condition ($k^2 = 0$) is imposed from outside by hand. This is not the case in our discussion because it is based on the BRST formalism where the Lagrangian density is gauge-fixed right from the beginning. In fact, for our description in the framework of the BRST formalism, the transversality condition (i.e. $k_\mu e^{\mu\nu}(k) = -e^{\mu\nu}(k) k_\nu = 0$) and the dual-transversality condition (i.e. $\varepsilon_{\mu\nu\sigma\zeta} e^{\nu\sigma}(k) k^\zeta = 0$) emerge from the physicality criteria w.r.t. the conserved and nilpotent (co-)BRST charges (cf. (4.21)). It is extremely gratifying to point out that the transformed polarization matrices (5.11) and (5.22) due to $T(2)$ subgroup of the Wigner's little group contain both the gauge and the dual-gauge transformations (5.4) and (5.13) for the specific choices and inter-relationships among some of the (dual-)gauge transformation parameters. In fact, the latter transformation parameters are chosen in terms of the parameters of the translation $T(2)$ subgroup of the Wigner's little group in a certain specific way (cf. (5.12), (5.16), (5.23), (5.24)) for the validity of our above claim. These statements can be recast and re-expressed in the language of BRST formalism where the (dual-)gauge transformation parameters are replaced by the anti-commuting (anti-)ghost fields. Another interesting feature, connected with the equations of motion for the fermionic vector (anti-)ghost fields (i.e. $\Box C_\mu = \frac{3}{2} \partial_\mu (\partial \cdot C), \Box \bar{C}_\mu = \frac{3}{2} \partial_\mu (\partial \cdot \bar{C})$) *is the normal mode expansions in (4.13).* It will be noted that the expansion in (4.13) can be true iff $\Box C_\mu = \Box \bar{C}_\mu = 0$ which imply that $(\partial \cdot C) = (\partial \cdot \bar{C}) = 0$. These conditions in the momentum phase space will correspond to $k^2 = 0, \; k \cdot C = k \cdot \bar{C} = 0$. It is nice to point out that these conditions are met throughout our discussions because of the choice of the 4-momentum vectors $k^\mu = (\omega, 0, 0, \omega)^T, k_\mu = (\omega, 0, 0, -\omega)^T$ and the conditions $\bar{C}_3 = -\bar{C}_0, C^3 = C^0$ (cf. (5.27)) on the (anti-)ghost fields which emerge due to the *equivalence* between the (dual-)gauge (or (co-)BRST) transformations on the polarization tensor $e^{\mu\nu}(k)$ and such type of transformations generated by the translational subgroup $T(2)$ of the Wigner's little group. Rest of the normal mode expansions in (4.13) are consistent with the equations of motion emerging from the Lagrangian density (3.12) that respects the on-shell nilpotent (co-)BRST symmetries. Thus, *the Wigner's little group plays a very decisive and crucial role in the correctness of the normal mode expansion in (4.13).*

For the 2D free Abelian one-form gauge theory, it is clear that the Wigner's little group becomes trivial and it generates *no* transformation on the polarization vector $e_\mu(k)$. By contrast, in the framework of the BRST cohomology, the (dual-)gauge transformations on $e_\mu(k)$ can be discussed elegantly which finally imply that this theory belongs to a new class of topological field theory (see, e.g., [10] for details). As far as the degrees of freedom count on $e^{\mu\nu}(k)$ for the 2-form gauge theory is concerned, it turns out that all the components of $e^{\mu\nu}(k)$ can be gauged away by exploiting the (co-)BRST symmetries *together*, treating them in an *independent* way. However, it has been shown [15] that the total (co-)BRST invariant Lagrangian density (3.12) does not represent an exact topological field theory. Rather, it presents an example of a quasi-topological field theory in the flat 4D spacetime (see, e.g., [15] for details) because the additional scalar fields ϕ_1, ϕ_2 and the ghost fields do exist in the theory besides the topological field $B_{\mu\nu}$. It would be nice to generalize our present work to the discussion of the higher rank anti-symmetric tensor fields in 4D and more than 4D of spacetime following the method adopted in [43]. Another direction that could be pursued is the discussion of the (dual-)gauge (or (co-)BRST) type symmetries for the case of the linearized gravity theory which has been recently studied in the framework of the Wigner's little group [44]. These are some of the issues that are under investigation and our results would be reported elsewhere in our future publications [45].

Note Added in the Proof: After our paper was accepted for publication, we came to know about a paper: H. Hata, T. Kugo and N. Ohta *Nucl. Phys.* **178**, 527 (1981) where BRST analysis has been performed, albeit in a different context, for the 2-form gauge field.

References

[1] P. A. M. Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School of Science (Yeshiva University Press, Yeshiva (N. Y.), 1964).

[2] For a review, see, e. g., K. Sundermeyer, *Constrained Dynamics*, Lecture Notes in Physics, vol. 169 (Springer-Verlag, Berlin, 1982).

[3] T. Eguchi, P. B. Gilkey and A. J. Hanson, *Phys. Rep.* **66**, 213 (1980).

[4] S. Mukhi and N. Mukunda, *Introduction to Topology, Differential Geometry and Group Theory for Physicists* (Wiley Eastern, New Delhi, 1990).

[5] J. W. van Holten, *Phys. Rev. Lett.* **64**, 2863 (1990).
J. W. van Holten, *Nucl. Phys.* B **339**, 258 (1990).

[6] K. Nishijima, *Prog. Theor. Phys.* **80**, 897 (1988).
K. Nishijima, *Prog. Theor. Phys.* **80**, 905 (1988).

[7] G. Fülöp and R. Marnelius, *Nucl. Phys.* B **456**, 442 (1995).
H. Aratyn, *J. Math. Phys.* **31**, 1240 (1990).

[8] R. P. Malik, *J. Phys. A: Math Gen* **33**, 2437 (2000) (*Preprint* hep-th/9902146).

[9] R. P. Malik, *Int. J. Mod. Phys.* A **15**, 1685 (2000) (*Preprint* hep-th/9808040).

[10] R. P. Malik, *J. Phys. A: Math Gen* **34**, 4167 (2001) (*Preprint* hep-th/0012085).

[11] R. P. Malik, *Mod. Phys. Lett.* A **15**, 2079 (2000) (*Preprint* hep-th/0003128).

[12] R. P. Malik, *Mod. Phys. Lett.* A **16**, 477 (2001) (*Preprint* hep-th/9711056).

[13] R. P. Malik, *Mod. Phys. Lett.* A **14**, 1937 (1999) (*Preprint* hep-th/9903121).

[14] E. Harikumar, R. P. Malik and M. Sivakumar,
J. Phys. A: Math Gen **33**, 7149 (2000) (*Preprint* hep-th/0004145).

[15] R. P. Malik, *J. Phys. A: Math Gen* **36**, 5095 (2003) (*Preprint* hep-th/0209136).

[16] R. P. Malik, *Phys. Lett.* B **521**, 409 (2001) (*Preprint* hep-th/0108105).

[17] R. P. Malik, *J. Phys. A: Math Gen* **35**, 3711 (2002) (*Preprint* hep-th/0106215).

[18] R. P. Malik, *Mod. Phys. Lett.* A **17**, 185 (2002) (*Preprint* hep-th/0111253).

[19] R. P. Malik, *J. Phys. A: Math Gen* **35**, 6919 (2002) (*Preprint* hep-th/0112260).

[20] R. P. Malik, *J. Phys. A: Math Gen* **35**, 8817 (2002) (*Preprint* hep-th/0204015).

[21] R. P. Malik, *Ann. Phys.* (N.Y.) **307**, 1 (2003) (*Preprint* hep-th/0205135).

[22] J. Thierry-Mieg, *J. Math. Phys.* **21**, 2834 (1980).
J. Thierry-Mieg, *Nuovo Cimento* A **56**, 396 (1980).

[23] M. Quiros, F. J. De Urries, J. Hoyos, M. J. Mazon and E. Rodrigues, *J. Math. Phys.* **22**, 767 (1981).

[24] L. Bonora and M. Tonin, *Phys. Lett.* B **98**, 48 (1981).
L. Bonora, P. Pasti and M. Tonin M, *Nuovo Cimento* A **63**, 353 (1981).

[25] L. Baulieu and J. Thierry-Mieg, *Nucl. Phys.* B **197**, 477 (1982).
L. Alvarez-Gaumé and L. Baulieu, *Nucl. Phys.* B **212**, 255 (1983).

[26] D. S. Huang and C. -Y. Lee, *J. Math. Phys.* **38**, 30 (1997).

[27] A. S. Schwarz, *Lett. Math. Phys.* **2**, 247 (1978).

[28] E. Witten, *Commun. Math. Phys.* **117**, 353 (1988).
E. Witten, *Commun. Math. Phys.* **121**, 351 (1989).

[29] See, e. g., for a review, D. Birmingham, M. Blau, M. Rakowski and G. Thompson, *Phys. Rep.* **209**, 129 (1991).

[30] S. Weinberg, *Phys. Rev.* B **134**, 882 (1964).

[31] S. Weinberg, *Phys. Rev.* B **135**, 1049 (1965).

[32] See, also, S. Weinberg, *Quantum Theory of Fields: Foundations* vol. 1 (Cambridge University Press, Cambridge, 1996).

[33] D. Han and Y. S. Kim, *Am. J. Phys.* **49**, 348 (1981).

[34] D. Han, Y. S. Kim and D. Son, *Phys. Rev.* D **26**, 3717 (1982).

[35] D. Han, Y. S. Kim and D. Son, *Phys. Rev.* D **31**, 328 (1985).

[36] R. Banerjee and B. Chakraborty, *Phys. Lett.* B **502**, 291 (2001).

[37] S. Weinberg, *Quantum Theory of Fields: Modern Applications* vol. 2 (Cambridge University Press, Cambridge, 1996).

[38] K. Nishijima, *Progress in Quantum Field Theory* (H. Ezawa and S. Kamefuchi, Eds.) (North Holland, Amsterdam, 1986);
For a review, see, e.g., K. Nishijima, *Czech. J. Phys.* **46**, 1 (1996).

[39] N. Nakanishi and I. Ojima, *Covariant Operator Formalism of Gauge Theories and Quantum Gravity* (World Scientific, Singapore, 1990).

[40] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton: New Jersey, 1996).

[41] S. Deser, A. Gomberoff, M. Henneaux and C. Teitelboim, *Phys. Lett.* B **400**, 80 (1997).

[42] I. J. R. Aitchison and A. J. G. Hey, *Gauge theories in Particle Physics: A Practical Introduction* (Adam Hilger, Bristol, 1982).

[43] See, e. g., V. O. Rivelles, *Revisita Brasileira de Fisica* **21**, 374 (1991).

[44] T. Scaria and B. Chakraborty, *Class. Quantum Grav.* **19**, 4445 (2002).

[45] R. P. Malik, in preparation.